Coherence function for the stochastic scattering by a
time-varying, slightly rough, acoustically soft surface

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Approximations for the coherence function of the random acoustic field scattered by a moving rough surface are extracted from a second-order perturbation expansion. The statistically rough surface is characterized by a simple, wide-sense stationary autocorrelation function that exhibits temporal and spatial dispersive behavior in terms of three parameters: the acoustically-small mean-square surface height, a correlation length, and a group velocity. Excitation is provided by the quintessential obliquely-incident time-harmonic plane-wave. Asymptotic evaluation of Fourier integral representations of the stochastic field yields expressions for the coherence function that explicitly display the dependence upon the space and time coordinates and their interaction via the geometrical parameters of the rough surface. © 2009 Acoustical Society of America. [DOI: 10.1121/1.3158927]

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I. INTRODUCTION

The fundamental importance and physical interpretation of the mutual coherence function of the acoustic field scattered by the time-varying, rough surface of the ocean are summarized by Parkins, who considered plane-wave excitation of a finite patch of surface. The familiar \( \sin x/x \) angular behavior due to such “finite aperture” sources is prevalent in all of his results. Clay and Medwin followed a similar course and focus on the coherence between partial field contributions scattered by separate subareas. Both Parkins and Clay and Medwin used a first-order correction to the Kirchhoff–Helmholtz integral. Harper and Labianca made serious analytic progress by applying perturbation theory to two different time-varying rough surfaces; one having a spatially uniform autocorrelation function and one comprised of a finite sum of deterministic, sinusoidal surface waves. The results of Harper and Labianca explicitly display the expected Doppler effects and additional complications due to an inhomogeneous (refractive) medium. Acoustically large surface roughness is treated by Dowling and Jackson using quadrature on the surface integral in the Kirchhoff or physical acoustics approximation. More in line with the philosophy of the present paper is the work of Shaw et al. who sought explicit, analytic expressions for the coherence functions of the scattered field by starting with reasonably simple, yet physically meaningful, rough surface correlation functions. They derived asymptotic expansions for the Kirchhoff integral. Unlike the present paper, their scattering surfaces do not vary in time. The important small parameter \( k_0d \) is the focus of the perturbation expansion, where \( k_0 = \omega_0/c \) is the wavenumber of the incident plane-wave and where \( a \) is the rms value of the random surface height.

When a time-harmonic plane-wave with time-dependence \( \exp(-i\omega_0 t) \) interacts with the time-varying rough surface depicted in Fig. 1, the spectrum of the scattered field is not restricted to the single excitation frequency \( \omega_0 \). However, in the specialized case where the autocorrelation function of the statistically stationary rough surface can arise from a temporally static surface in uniform translation, simple coordinate transformation of well known static results from perturbation theory is applicable. The acoustic pressure field \( \psi(x, y, t) \) is a function of the two spatial coordinates \( x \) and \( y \), the time \( t \), and is a solution to the source-free scalar wave equation. The Dirichlet boundary condition \( \psi(x, a\xi(x, t), t) = 0 \) applies to the total field, for all time, on the soft surface \( y_r = a\xi(x, t) \). The scattered field, which is defined here as the difference between the total field and the incident plane-wave that exists everywhere in the absence of the impenetrable surface, exhibits radiative behavior as \( y \rightarrow +\infty \).

By expressing the soft surface as \( y_r = a\xi(x, t) \), where the rms surface height \( a \) appears explicitly, the mean-square value of the normalized random surface \( \xi(x, t) \) is unity. This real-valued two-dimensional stochastic process (or field) is assumed to be wide-sense stationary in both its spatial \((x)\) and temporal \((t)\) dependencies. It has zero mean-value \( E[\xi(x, t)] = 0 \), and the specific autocorrelation function adopted for the present study is

\[
E[\xi(x, t)\xi(x', t')] = R(x - x', t - t')
= \exp\left\{-\frac{(x - x')^2 - u(t - t')^2}{b}\right\},
\]

This is a simpler version of a surface autocorrelation function proposed by Medwin and Clay. Initial studies that incorporated their additional cosine factor were complex enough to warrant a reduction in the number of parameters. The stochastic scattered field caused by a rough surface described by the simple function (1) is rich enough in physical

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features to capture our present attention while it maintains the dominant features of the time-varying rough surface. The parameter $u$ (m/s) is a “group velocity” in that the Gaussian function in Eq. (1) propagates in the $x$ direction with speed $u$. The parameter $b$ (m) is a correlation length of the random surface.

A time-harmonic plane-wave of unit amplitude and frequency $\omega_0$ is incident from the angle $\phi_i$ measured from grazing. In the absence of the roughness, that is, for $a=0$, the total field is the sum of the incident and reflected plane waves

$$\psi_0(x,y,t) = e^{-ik_0(x\cos\phi_i+ct)}[e^{-ik_0y}\sin\phi_i - e^{ik_0y}\sin\phi_f] = -2i\sin(k_0y)\sin\phi_i e^{-ik_0(x\cos\phi_i+ct)},$$

where $k_0=\omega_0/c$ is the exciting wavenumber with $c$ the sound speed. This is the zeroth-order or geometrical acoustics field. The common perturbation expansion for the total field in terms higher in order than $k_0$ is

$$\psi(x,y,t) = \psi_0(x,y,t) + k_0a \psi_1(x,y,t) + (k_0a)^2 \psi_2(x,y,t) + \cdots.$$  

Terms higher in order than $(k_0a)^2$ are an unnecessary complication because the case of small $k_0a$ is sufficiently interesting and manageable. Perturbation theory and the philosophy of the second-order correlation approach to the stochastic problem are nicely treated by Rytov et al.\(^8\) and by Sobczyk.\(^9\) Interpretation and dynamic evolution of the coherence function are detailed by Beran and Parrent.\(^9\) Warnick and Chew\(^10\) gave a condensed summary of the voluminous literature on various approaches to rough surface scattering: An observation of the importance of analytic methods in building physical insight is still true. The paper by Shaw et al.\(^5\) is similar in spirit to the work here, except that their use of finite source distances requires the evaluation of particular integrals. The uniform plane-wave is nice because finite source effects do not clutter the results.

**II. FIRST-ORDER PERTURBATION FIELD**

The first-order perturbative field is forced by the non-zero boundary data

$$\psi_1(x,0,t) = \frac{\zeta(x,t)}{k_0} \frac{\partial}{\partial y} \psi_0(x,0,t) = 2i \sin(\phi_i) \zeta(x,t) e^{-ik_0(x\cos\phi_i+ct)},$$

and it inherits its zero mean $E[\psi_1(x,y,t)]=0$ from that of $\zeta(x,t)$. The autocorrelation function of the complete $\psi_1(x,y,t)$ can be worked out from scratch using multiple Fourier integrals in all of the independent variables. However, because the autocorrelation function (1) of the rough surface can arise as a simple translation

$$E[\zeta(x-ut)\zeta(x'-ut')] = R((x-x')-u(t-t')),$$

where

$$E[\zeta(x)\zeta(x')] = R(x-x')$$

defines the wide-sense statistical stationarity of a temporally static surface, a proper coordinate transformation as given by Morse and Ingard,\(^11\) for example, can be applied directly to the standard results of perturbation theory for the static surface. These are summarized by Sobczyk,\(^8\) for example. When applied to a Fourier integral representation for the first-order scattered field, the coordinate transformation exhibited in Eq. (5) gives

$$E[\psi_1(x,y,t)\psi_1^*(x',y',t')]
= \frac{2b}{\sqrt{\pi}} \sin^2 \phi_i \exp \left[ -i\omega_0 \left( 1 + \frac{u}{c} \cos \phi_i \right) (t-t') \right]
\cdot \int_{-\infty}^{\infty} d\alpha \exp \left[ -\frac{1}{4} \left( \alpha + k_0 \cos \phi_i \right)^2 \right]
\cdot \exp[i(\alpha(x-x') + \beta y - \beta^* y' - u\alpha(t-t'))].$$

The separation constant $\beta$ that accompanies the $y$-dependence, or the $y$-component of the wavenumber, is defined by the usual separation equation

$$\alpha^2 + \beta^2 = k^2,$$  

but now

$$k = k_0 + \frac{u}{c}(\alpha + k_0 \cos \phi_i).$$

The coherence function of $\psi_1(x,y,t)$ is wide-sense stationary in $x$ and $t$ since only the differences $x-x'$ and $t-t'$ appear in Eq. (7). The evanescent contributions that manifest when $\beta$ is imaginary are nonstationary in $y$. But for both $y$ and $y'$ sufficiently large, such exponentially decreasing contributions to the integral are negligible and the first-order perturbation field is effectively wide-sense stationary in all three of the independent variables $x$, $y$, and $t$.

The demarcation between evanescence and propagation, according to Eq. (8), occurs when $k^2=\alpha^2$. The two solutions to this quadratic are

$$\alpha = \frac{k_0 \left( 1 + \frac{u}{c} \cos \phi_i \right) \pm \sqrt{1 - \frac{u}{c}}}{1 - \frac{u}{c}}$$

FIG. 1. Plane-wave incident upon a time-varying, rough surface.
Hence, under the restriction \( y = y' \) big enough to neglect the nonstationary evanescent contributions, the “far-field” coherence function of the first-order perturbation field is

\[
E\{\psi_1(x,y,t)\psi_1^*(x',y',t')\} = 2 \sin^2 \phi_i \exp \left[ -i \omega_0 \left( 1 + \frac{u}{c} \cos \phi_i \right) (t - t') \right]
\cdot \exp \left[ -i k_{0s} \cos \phi_i \exp \left[ - \left( \frac{s}{b} \right)^2 \right] \right] \left[ \text{erf}(B) - \text{erf}(A) \right].
\]  

(19)

The error function of complex argument is computed using the algorithm of Weideman.\textsuperscript{12} The resultant specialized \( y = y' \gg 1 \) coherence function (12) explicitly displays the dependence upon all of the geometrical and acoustical parameters of the physical problem.

For large values of the independent variable \( s \to \infty \), a growth factor in the error functions tends to cancel the rapid decay in the Gaussian function. Calculations and insight are improved by a large \( s \)-expansion. It is more efficient to work directly with the original integral \( I \) rather than grappling with the asymptotics of the separate error functions. A single integration-by-parts applied to Eq. (14) yields a single-term asymptotic expansion

\[
I(s) = \int_{a}^{b} d\alpha f(\alpha) e^{i\alpha s}
= \left. \frac{f(\alpha)e^{i\alpha s}}{is} \right|_{a}^{b} \cdot I(s).
\]  

(20)

Algebra that exploits the details exhibited by Eqs. (17) and (18) yields the large \( s \) form

\[
isI(s) = \exp \left[ - \left( \frac{k_{0b}}{2} \cos^2 \frac{1}{2} \phi_i \right) \frac{(1 + \frac{u}{c})}{(1 - \frac{u}{c})} k_{0s} \right] \cdot \exp \left[ - \left( \frac{k_{0b}}{2} \sin^2 \frac{1}{2} \phi_i \right) \frac{(1 + \frac{u}{c})}{(1 - \frac{u}{c})} k_{0s} \right].
\]  

(21)

Note that the magnitude of the associated coherence function ultimately decays as \( |s|^{-1} \).

III. SECOND-ORDER PERTURBATION FIELD

The second-order field \( \psi_2(x,y,t) \) in Eq. (3) is forced by the boundary data from the first-order field, as in

\[
\psi_2(x,0,t) = \frac{\xi(x,t)}{k_0} \frac{\partial}{\partial y} \psi_1(x,0,t).
\]  

(22)

A single integral representation for the expectation of the second-order perturbation field is

\[
E\{\psi_2(x,y,t)\psi_2^*(x',y',t')\} = \frac{\partial}{\partial y} \psi_1(x,0,t)
\cdot \int_{-\infty}^{\infty} d\alpha d\beta e^{i\beta y} \left[ - \left( \frac{1}{4} \left( \alpha + k_{0s} \cos \phi_i \right) \right)^2 \right],
\]  

(23)

where Eqs. (8) and (9) still apply. The \( x \) and \( t \) behavior is
identical to that of the incident plane-wave. A stationary
phase approximation of the integral, as \( y \to \infty \), yields

\[
E\{\psi_2(x,y,t)\} = \frac{k_0 b}{\sqrt{2k_0 y}} \sin \phi_t \frac{\left(1 + \frac{u}{c} \cos \phi_t\right)^{3/2}}{\left[1 - \left(\frac{u}{c}\right)^2\right]^{1/4}} \\
\times \exp\left\{-\frac{1}{4} \left(k_0 b \frac{u}{c} + \cos \phi_t\right)^2 \left[1 - \left(\frac{u}{c}\right)^2\right]^{1/2}\right\} \cdot \exp\left[-i(k_0 y \cos \phi_t + k_0 c t + \pi/4)\right] \cdot \\
\exp\left\{i k_0 y \frac{1 + \frac{u}{c} \cos \phi_t}{1 - \left(\frac{u}{c}\right)^2}\right\},
\] (24)

**IV. DISCUSSION OF RESULTS**

To order \((k_0 a)^2\) the mean value of the perturbation expansion (3) is

\[
E\{\psi(x,y,t)\} = \psi_0(x,y,t) + (k_0 a)^2 E\{\psi_2(x,y,t)\},
\] (25)

where \(\psi_0\) is the deterministic, geometrical acoustics field (2) and where the large \( y \) behavior of the mean value of \(\psi_2\) is given by Eq. (24). Recall from the development that the mean value of \(\psi_t\) is zero. As \( u/c \to 1^- \) the stationary phase approximation of the integral in Eq. (23) requires adjustment, but for reasonable values of normalized speed, the information in the approximation (24) is substantial. This perturbation term in the distant mean field (25) is proportional to \( y^{-1/2} \) and it inherits the \( x \) and \( t \) behavior of the incident wave. The Gaussian factor in Eq. (24) shows that \( E\{\psi_2\} \) rapidly dissipates with the important parameter \( k_0 b \). Large \( k_0 b \) is the high frequency regime where the correlation length \( b \) of the rough surface interacts with a substantial number of the incident acoustic wavelengths, with the net effect that the superposition of the random contributions to the scattered field tends to cancel on the average. However, if \( k_0 b \) is small enough so that the Gaussian factor is closer to unity, then the leading \( k_0 b \) in Eq. (24) becomes the primary limiting factor. The strength of the specular component \( \psi_2 \) is a strong function of the incidence angle \( \phi_t \). In the more interesting case of large \( k_0 b \), the amplitude factor in Eq. (24) becomes highly peaked around its maximum at \( \cos \phi_t = -u/c \), where, in an average sense, the incident wavefront is essentially keeping up with the surface fluctuations.

The autocorrelation or coherence function of the total field is, again up to order \((k_0 a)^2\),

\[
E\{\psi(x,y,t) - E\{\psi(x,y,t)\}\} |\psi_0^{*}(x',y',t') - E\{\psi_0^{*}(x',y',t')\}\} = (k_0 a)^2 E\{\psi_0(x,y,t) \psi_0^{*}(x',y',t')\}.
\] (26)

Hence, the coherence function (7) of the first-order perturbation field is the central descriptor of the scattering from the rough surface. With the independent space/time variable \( s \)

**FIG. 2.** Coherence function of first-order perturbation field. Case: \( u/c=0 \) and \( \phi_t=\pi/4; s=x-x'-u(t-t') \) and \( y=y' \gg 1 \).

defined in Eq. (13), introduce the abbreviated notation

\[
|\rho(s/b)| = \left| E\{\psi_1(x,y,t) \psi_0^{*}(x',y',t')\} \right|
\] (27)

for the magnitude of the large \( y=y' \) coherence function. The grouping of the field coordinates \( x, x', t, \) and \( t' \) occurs naturally in the physical mathematics, as in Eq. (13), and therefore we follow nature’s lead and keep these coordinates together as they appear. It is entirely expected that the ratio \( s/b \) of the appropriately organized space-time coordinates to the rough surface correlation length is the important parameter in the scattering problem.

The variation in \( |\rho| \) with normalized \( s/b \) is graphed in Figs. 2–4 for \( u/c=0, 0.5, \) and 0.9, respectively, when the

**FIG. 3.** Coherence function of first-order perturbation field. Case: \( u/c=0.5 \) and \( \phi_t=\pi/4; s=x-x'-u(t-t') \) and \( y=y' \gg 1 \).
exciting plane-wave is incident from the representative angle \( \phi_0 = \pi/4 \). The three curves in each of Figs. 2–4 correspond to low, intermediate, and high frequencies as far as the correlation length \( b \) is concerned. The selected values \( k_0b = 1/10, 1, \) and 10 cover a wide range of frequencies.

Figure 2 shows that when the rough surface group velocity is zero, the first-order coherence function is a rather peaked function of \( s/b \) for the high frequency \( (k_0b = 10) \), compared to the essentially flat variation at the low frequency \( (k_0b = 0.1) \), that is substantially lower (about 5%) in magnitude. As \( u/c \) is increased, first to 0.5 in Fig. 3 and then to 0.9 in Fig. 4, the lower frequency graphs of \(|\rho|\) display behavior more like that of the higher frequency \(|\rho|\) from the static \( u/c = 0 \) surface. A faster group velocity \( u \) in the rough surface correlation function (4) lets the incident acoustic wave experience more interactions with the roughness per wavelength. Therefore, if the acoustic correlation length \( k_0b \) is relatively small, the effect of increasing the group velocity \( u/c \) is similar to starting with a larger correlation length. It is “per wavelength” because both the exciting plane-wave and the scattering surface are infinite in extent, and the natural length scale that has physical meaning for the wave is its inherent spatial period.

This feature, that is so apparent in the graphs of Figs. 2–4, of increasing \( u/c \) mimicking a larger \( k_0b \), is also obvious in the mathematics. The effect is clearly seen in the arguments \( A \) and \( B \), defined in Eqs. (17) and (18), respectively, of the error functions in Eq. (19). The physical problem of Fig. 1 is symmetric about \( \phi_0 = \pi/2 \), allowing for the sign of \( u/c \). If \( u/c \) approaches +1, or simply increases positively, then the effect is the same as increasing \( k_0b \) in the parameter \( B \) of Eq. (18). Similarly, if \( u/c \) approaches −1, then the parameter \( A \) of Eq. (17) experiences an effectively larger \( k_0b \).

The correlation function (19) and the graphical display of its behavior in Figs. 2–4 is the product of the analysis reported here and is, of course, specific to the assumed surface correlation function (1). Although rather restricted and idealized, such analytical results build insight into the stochastic scattering physics and are potentially useful in interpreting statistical data generated by Monte Carlo simulation using purely numerical models.

V. CONCLUSIONS

The coherence function for the random field scattered by the time-varying, slightly-rough surface is proportional to \( (k_0a)^2 \), where \( k_0 = o/c \) is the wavenumber of the incident acoustic wave and \( a \) is the rms value of the rough surface. The scattered field is wide-sense stationary in both time and in the coordinate that is parallel to the rough surface. Far from the surface, the scattered field is also approximately wide-sense stationary in the normal coordinate. Two important parameters are \( k_0b \), where \( b \) is the correlation length of the rough surface, and the normalized group velocity \( u/c \). If the surface group velocity is zero and if \( k_0b \) is small, say, \( k_0b \leq 1 \), the random component of the scattered field is diffuse and rather uninteresting. Acoustically large correlation lengths, such as \( k_0b \geq 10 \), produce a scattered field that is significantly correlated over a surface correlation length \( b \), regardless of the surface group velocity. If \( k_0b \) remains small but the surface group velocity becomes a sizable fraction of the acoustic velocity, say, \( u/c \geq 1/2 \), then the scattered field is noticeably imprinted with the dispersive characteristics of the rough surface and the correlation function maintains its cohesion and propagates similar to the case of larger \( k_0b \).

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