The complete extension of the Biot–Tolstoy solution to the density contrast wedge with sample calculations

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The Biot–Tolstoy exact time-domain solution for the three-dimensional impulse response of an impenetrable wedge is extended to accommodate the isovelocity or density-contrast wedge. Fourier transformation of the time and axial variables, along with a Kantorovich–Lebedev transform applied to the cylindrical radial coordinate, leads to a solution in terms of residue series. When the wedge angle is a rational fraction of \(\pi\), the residue series can be reduced to a finite sum which is evaluated for some special cases. The total pressure field consists of geometrical acoustics contributions, as predicted by Snell’s laws, plus a modified version of the Biot–Tolstoy diffraction field. © 1997 Acoustical Society of America. [S0001-4966(97)07004-5]

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INTRODUCTION

The three-dimensional problem of Fig. 1 is the scattering of the acoustic field radiated by an impulsive point source in the presence of a penetrable wedge having the same sound velocity as the external medium. The exact solution to this canonical boundary value problem is a first step toward extending some widely used techniques, notably the wedge-assemblage method\(^1\) for rough surface scattering and the related variants of the geometrical theory of diffraction, to model scattering by penetrable surfaces. In underwater acoustics, the ratio of wave speeds in sea water (external medium) and in the ocean floor (internal medium) is often close to unity, i.e., \(0.9 < c_e/c_i < 1.2\) for a variety of ocean floor sediments. This is the primary motivation for the present density-contrast solution.

Chu\(^2\) suggested how the Biot–Tolstoy solution\(^3\) might be extended to include the phenomenon of penetration when only a density contrast exists between the interior and exterior media but gave an incomplete presentation. He was unable to relate his coefficients to the source strength and considered only the known limiting cases. The following work first constructs the time-domain solution for an arbitrary wedge angle by the use of appropriate integral transforms and representations. This calculation differs principally from the impenetrable case in the nonavailability of the closed-form residue sum expression that subsequently yields a solution in a periodic form that ultimately allows each series to be reduced to a finite number of terms. This is accomplished by writing the wedge angle in the special form \(\alpha = n\pi/M\). The ensuing analysis takes full advantage of symmetries in the eigenvalue equation for the required poles, and hence considers two separate classes of wedge angles: The case having \(M\) and \(N\) both odd in Sec. IV and the case of either \(M\) or \(N\) even in Sec. V. The explicit total field is presented in Sec. VI for the example cases \(\alpha = \pi/3, 2\pi/5, \pi/4,\) and \(\pi/6\). The results of Sec. VI clearly show the expected, but nontrivial, modifications to the geometrical
acoustics part of the total field due to penetration. Sec. VII is a detailed compilation and study of the diffraction field as the reflection coefficient $\Gamma$ is varied from $-1$ (the perfectly soft, impenetrable case) to the other extreme of $+1$ (the perfectly hard, impenetrable case).

The dominant effects of a small velocity contrast can now be included, in principle, by a perturbative adjustment to this zero-order or isovelocity solution. Continuing progress based on the results presented here is planned via the adaptation of an iterative scheme by Rawlins,\(^4\) valid for refractive indices in the range 1 to $\sqrt{2}$. Exploitation of the similar wave speeds circumvents the great difficulties inherent in an infinite boundary between arbitrarily different media.\(^5,6\) The scattered waveforms resulting from more complicated source distributions, in both space and time, are obtained immediately by an appropriate spatial and/or temporal convolution between the source function and the resultant impulse response or Green’s function of this paper.

I. PENETRATION PROBLEM WITH IMPULSIVE POINT SOURCE

The scalar pressure field $p(r, \phi, z, t)$ due to an impulsive source at $(R, \Phi, 0, 0)$ external to a wedge of angle $2\alpha$ having the same wave speed $c$ as the surrounding medium (Figs. 1 and 2) is the solution to

$$\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) p^{(c)}(r, \phi, z, t) = \frac{\delta(r-R)}{R} \delta(\phi-\Phi) \delta(z) \delta(t)$$

$$(\alpha < \phi < \pi, \alpha < \phi < 2\pi - \alpha),$$

$$\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) p^{(i)}(r, \phi, z, t) = 0 \quad (-\alpha < \phi < \alpha),$$

subject to the boundary conditions

$$p^{(c)} = p^{(i)}$$

and

$$\rho \frac{\partial p^{(c)}}{\partial \phi} = \rho_0 \frac{\partial p^{(i)}}{\partial \phi}$$

at $\phi = \pm \alpha$, \hspace{1cm} (2)

where the density ratio is $\rho = \rho_0/\rho_1$. Fourier transformation in both $t$ and $z$, according to

$$\hat{p}(r, \phi, l, \omega) = 2 \int_{-\infty}^{\infty} e^{-i \omega t} \int_{-\infty}^{\infty} p(r, \phi, z, t) \cos lz \, dz \, dt$$

and the definitions $k = \omega/c$, $\kappa = \sqrt{k^2 - \kappa^2}$, result in

$$\left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + \kappa^2 \right] \hat{p}^{(c)}(r, \phi, l, \omega) = \frac{\delta(r-R)}{R} \delta(\phi - \Phi),$$

$$\left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + \kappa^2 \right] \hat{p}^{(i)}(r, \phi, l, \omega) = 0.$$\hspace{1cm} (4)

Before proceeding further, it is appropriate to emphasize the difficulties due to penetration by noting that in the hard ($\rho = \infty$) or soft ($\rho = 0$) case, only the external problem is relevant and can be solved by introducing image singularities. The periodic delta function, defined by

$$\delta(\theta) = \sum_{n=-\infty}^{\infty} \delta(\theta - n \pi) = \frac{1}{\pi} \left[ \frac{1}{2} + \sum_{m=1}^{\infty} \cos m \theta \right],$$\hspace{1cm} (5)

enables this to be achieved by replacing $\delta(\phi - \Phi)$ in (4) by...
Thus, for \( v > 0 \),

\[
\hat{p}^{(c)}(r, \phi, l, \omega) = \frac{\pi i}{2(\pi - \alpha)} \left\{ \begin{array}{c}
\frac{1}{2} \\
0
\end{array} \right\} H^{(2)}_0(\kappa r) J_0(\kappa r)
= \frac{\pi i}{2(\pi - \alpha)} \left\{ \begin{array}{c}
\frac{1}{2} \\
0
\end{array} \right\} 
+ \sum_{m=1}^{\infty} H^{(2)}_m(\kappa r) J_m(\kappa r)
\times \left[ \frac{\cos v_m(\phi - \alpha)}{\sin v_m(\phi - \alpha)} \right] \left[ \frac{\cos v_m(\Phi - \alpha)}{\sin v_m(\Phi - \alpha)} \right],
\]

where

\[ v_m = \frac{m \pi}{2(\pi - \alpha)} \text{ and } \left( \begin{array}{c}
r > \\
r \
\end{array} \right) = \max (r, R). \]

The complex conjugate eigenfunction expansion is used for \( \omega < 0 \). However, this method is unavailable for the density contrast \((0 < \rho < \infty)\) considered here, and an appropriate integral transform is needed to determine \( \hat{p}^{(c)} \). In anticipation of this calculation, it is advantageous to work with the symmetric \( p_s \) and antisymmetric \( p_a \) parts of the pressure \( p \) in order to restrict the field to \( 0 < \phi < \alpha \) and \( \alpha < \phi < \pi \) in the interior and exterior regions, respectively.

**II. SYMMETRIC AND ANTISYMMETRIC PARTS OF THE SOLUTION**

Consider pairs of in-phase or out-of-phase singularities of half-strength, arranged symmetrically about the wedge bisector (Fig. 3). The source plus positive image is equivalent (in the upper half-space) to the source above a hard bisecting plane, while the antisymmetric arrangement of source plus negative image satisfies the soft boundary condition on the \( y = 0 \) plane. Upon solving for the symmetric and antisymmetric fields, the sum \( p_s + p_a \) is the solution of the wedge problem at a point on the same side of the centerline as the source. Meanwhile, the difference \( p_s - p_a \) gives the solution of the wedge problem at a point on the opposite side of the centerline from the source (Fig. 4).

Fourier transforms \( \hat{p}_s \) and \( \hat{p}_a \), defined by (3), satisfy according to (4), the wave equations

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + \kappa^2 \hat{p}^{(c)}(r, \phi, l, \omega) = \frac{\delta(r-R)}{2R} \delta(\phi-\Phi),
\]

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + \kappa^2 \hat{p}^{(a)}(r, \phi, l, \omega) = 0.
\]

The Kantorovich–Lebedev transform

\[
P_{s,a}(v, \phi, \kappa) = \int_0^\infty \hat{p}_{s,a}(r, \phi, \kappa) r^{-1} H^{(2)}_v(\kappa r) dr
\]

renders the pair of ordinary differential equations

\[
\frac{d^2}{d\phi^2} + \nu^2 \right) P^{(c)}_{s,a}(v, \phi, \kappa) = \frac{1}{2} H^{(2)}_v(\kappa R) \delta(\phi-\Phi) (\alpha < \phi < \pi),
\]

\[
\frac{d^2}{d\phi^2} + \nu^2 \right) P^{(a)}_{s,a}(v, \phi, \kappa) = 0 (0 < \phi < \alpha),
\]
subject to continuity boundary conditions at the media interface
\[ P_{s,a}^{(c)} = P_{s,a}^{(i)} \quad \text{and} \quad \rho \frac{dP_{s,a}^{(c)}}{d\phi} = \frac{dP_{s,a}^{(i)}}{d\phi} \quad \text{at} \quad \phi = \alpha \]  
and appropriate Neumann or Dirichlet conditions on the geometrical plane of symmetry
\[ \frac{dP_s}{d\phi} = 0 \quad \text{or} \quad P_s = 0 \quad \text{at} \quad \phi = 0, \pi. \]  
Boundary conditions at \( \phi = 0, \pi \) are satisfied by
\[ \frac{2\nu}{H_v(2)(\kappa R)} P_{s,a}^{(i)} = A_{\sin} \nu \phi \quad (0 < \phi < \alpha) \]  
and
\[ \frac{2\nu}{H_v(2)(\kappa R)} P_{s,a}^{(c)} = B_{\sin} \nu(\pi - \phi) + H(\Phi - \phi) \times \sin \nu(\Phi - \phi) \quad (\alpha < \phi < \pi), \]  
where \( H \) denotes the Heaviside unit function. Conditions (12) at \( \phi = \alpha \) then yield
\[ \hat{p}(r, \phi, \kappa) = \lim_{\nu \to 0} \frac{1}{4} \int_{-\infty}^{\infty} \exp(\nu^2)J_\nu(\kappa r)H_v(2)(\kappa R)f(\nu, \phi) d\nu \]  
\[ = \lim_{\nu \to 0} \frac{1}{4\pi i} \int_{-\infty}^{\infty} \exp(\nu^2)J_\nu(\kappa r) \int_{0}^{\gamma} \exp \left[ \frac{1}{2} \left( \frac{s - R^2 + \nu^2}{s} - \kappa^2 \right) \right] I_\nu \left( \frac{\kappa^2 r s}{s} \right) ds \, d\nu \]  
\[ = -\lim_{\nu \to 0} \frac{1}{8\pi i} \int_{-\infty}^{\infty} \exp(\nu^2)J_\nu(\kappa r) \int_{-\infty}^{\infty} e^{-r w} H_v(2)(\kappa^2 r^2 - 2 \kappa R \cosh w) w_{1/2} d\nu \, dw \]  
\[ = \frac{1}{4} \int_{-\infty}^{\infty} \left[ \kappa(r^2 + R^2 - 2 \kappa R \cosh w) w_{1/2}^2 \right] \sum_{n=0}^{\infty} \left[ \text{residues of } e^{-r w}f(\nu, \phi) \right. \at \nu = \nu_n \]  
dw \]  
The \( w \) integration is along the contour \( C \) of Fig. 5, \( \nu_0 = 0, \nu_n > \nu_{n-1} \quad (n \gg 1), \) and the prime on the summation denotes that a \( 1/2 \) factor multiplies the residue at the origin, if it exists. The result, symmetric in \( r,R, \) is also valid for \( r > R \) and the details are briefly described by Jones,\(^7\) after correcting some misprints. Inversion of the \( z \) and \( t \) transforms then yields
\[ p(r, \phi, z,t) = \frac{i}{4\pi} \int_{-\infty}^{\infty} \frac{\delta[ t - (1/c)(r^2 + R^2 - 2 \kappa R \cosh w + z^2)^{1/2}]}{(r^2 + R^2 - 2 \kappa R \cosh w + z^2)^{1/2}} \cdot \sum_{n=0}^{\infty} \left[ \text{residues of } e^{-r w}f(\nu, \phi) \right. \at \nu = \nu_n \]  
dw \]  
and subsequent deformation of the contour shows that
\[ A = \frac{(1 + \Gamma) \cos \nu(\pi - \Phi)}{\sin \nu \pi \pm \Gamma \sin \nu(\pi - 2\alpha)} \]  
and
\[ B = \frac{\cos \nu(\Phi - \alpha)}{\sin \nu(\pi - \alpha)} \quad (\rho = 0, \Gamma = -1), \]  
for which the eigenvalues are determined by (8). Even/odd values of \( m \) correspond to symmetric/antisymmetric solutions in the hard case but vice versa in the soft case. The \( \alpha = 0 \) limits of the symmetric hard case and antisymmetric soft cases must be identical to the symmetric and antisymmetric solutions at \( \Gamma = 0 \quad (\rho = 1). \) With the function \( f \) defined by
\[ f(\nu, \phi) = \frac{2\nu}{H_v(2)(\kappa R)} P(\nu, \phi, \kappa), \]  
inversion of the Kantorovich–Lebedev transform proceeds, for \( r < R, \) as
\[ \Gamma = \frac{\rho - 1}{\rho + 1} \quad (|\Gamma| \leq 1). \]  
In the two limiting cases
\[ B = \frac{\cos \nu(\Phi - \alpha)}{\sin \nu(\pi - \alpha)} \quad (\rho \rightarrow \infty, \Gamma = 1), \]  
\[ H_v(2)(\kappa R) \]
\[ p(r, \phi, z, t) = -\frac{1}{2\pi} \frac{c}{r \sin \eta} \sum_{n=0}^{\infty} \left[ \text{residues of } \cos \nu \eta \ f(\nu, \phi) \text{ at } \nu = \nu_n \right] \]

\[ + \frac{1}{2\pi} \frac{c}{r R \sinh \xi} \sum_{n=1}^{\infty} \left[ \text{residues of } e^{-\nu \xi} \sin \nu \pi \ f(\nu, \phi) \text{ at } \nu = \nu_n \right], \tag{21} \]

where the time and space dependence is folded into

\[ \cos \eta = \frac{R^2 + r^2 + \zeta^2 - c^2 t^2}{2R}, \tag{22} \]

\[ \cosh \xi = \frac{c^2 t^2 - R^2 - r^2 - \zeta^2}{2R}, \]

with \(0 < \eta < \pi\) and contributions occurring in relevant time intervals.

It is instructive at this stage to illustrate how the subsequent calculation might proceed by considering one of the simple impenetrable cases. In the soft case \((\Gamma = -1)\) the transform is, from (14) with \(e\) now disposable,

\[ f_{s, a}(\nu, \phi) = -\frac{\sin \nu (\Phi - \alpha) \cos \nu (\pi - \phi) \cos \nu (\pi - \phi)}{n \sin \nu (\pi - \alpha)}, \tag{23} \]

with \(\Phi\) and \(\phi\) interchanged if \(\phi < \Phi\). Poles for the symmetric and antisymmetric fields are

\[ \nu^s = \frac{(m + 1/2) \pi}{\pi - \alpha} \quad (m \geq 0) \quad \text{and} \quad \nu^a = \frac{m \pi}{\pi - \alpha} \quad (m \geq 0). \tag{24} \]

The first residue sum above for the total pressure \(p\) is now

\[ \sum_{n=0}^{\infty} \left[ \text{residues of } \cos \nu \eta \ f(\nu, \phi) \text{ at } \nu = \nu_n \right] \]

\[ = \frac{1}{\pi - \alpha} \sum_{n=0}^{\infty} \left[ \sin \frac{n \pi}{2(\pi - \alpha)} (\Phi - \alpha) \sin \frac{n \pi}{2(\pi - \alpha)} (\phi - \alpha) \cos \frac{n \pi \eta}{2(\pi - \alpha)} \right] \]

\[ = \frac{\pi}{4(\pi - \alpha)} \left[ \delta \left[ \frac{\pi}{2(\pi - \alpha)} (\Phi - \phi + \eta) \right] + \delta \left[ \frac{\pi}{2(\pi - \alpha)} (\Phi - \phi - \eta) \right] \right] \]

\[ - \delta \left[ \frac{\pi}{2(\pi - \alpha)} (\Phi + \phi - 2\alpha + \eta) \right] - \delta \left[ \frac{\pi}{2(\pi - \alpha)} (\Phi + \phi - 2\alpha - \eta) \right] \tag{25} \]

which, for \(\alpha < \phi, \Phi < \pi\), and \(0 < \eta < \pi\), reduces to the geometrical acoustics field

\[ \frac{1}{2} \left[ \delta(\eta - |\Phi - \phi|) - \delta(\eta + 2\alpha - \Phi - \phi) \right]. \tag{26} \]

The diffracted portion of the total field derives from the second residue sum of (21):

\[ \sum_{n=1}^{\infty} \left[ \text{residues of } e^{-\nu \xi} \sin \nu \pi \ f(\nu, \phi) \text{ at } \nu = \nu_n \right] \]

\[ = \frac{1}{2(\pi - \alpha)} \sum_{n=1}^{\infty} \left[ \cos \frac{n \pi}{2(\pi - \alpha)} (\Phi - \phi) - \cos \frac{n \pi}{2(\pi - \alpha)} (\Phi + \phi - 2\alpha) \right] \sin \frac{n \pi^2}{2(\pi - \alpha)} \exp \frac{-n \pi \xi}{2(\pi - \alpha)} \tag{27} \]

which can be simplified by using

\[ \sum_{n=1}^{\infty} e^{-n\beta} \sin n \gamma \sin \frac{\gamma}{2(\cosh \beta - \cos \gamma)}. \tag{28} \]

Substitution of (26), (27) in (21) then shows that the total field is given by
which is equivalent to the form given by Biot and Tolstoy\(^3\) and displays the features outlined in the introduction because of the time delay, given by (22), before the \(\xi\) variable becomes operative. This derivation of (29) is more efficient than that given by the original authors. Note the sign of the forcing term in (1).

However, the corresponding calculation in the penetrable case is substantially more difficult, if not impossible. For \(0<\rho<\infty\), the \(\phi\)-independent factor \(B\) of (15) is rewritten as

\[
B = \sin \frac{\nu(\pi-\Phi)}{\nu \pi \pm \Delta} \cos \nu(\pi-2\alpha) \cos \nu(\pi-2\alpha) \sin \nu(\pi-2\alpha) \sin \nu(\pi-\Phi),
\]

which allows the exterior version of (18) to be expressed

\[
f(\nu,\alpha) = \cos \nu \pi \pm \Delta \cos \nu(\pi-2\alpha) \cos \nu(\pi-2\alpha) \sin \nu(\pi-2\alpha) \sin \nu(\pi-\Phi) \times \cos \frac{\nu(\pi-\phi)}{\sin \nu \pi \pm \Delta} \sin \nu(\pi-\phi) + \frac{\sin \nu(\pi-\phi,-) \cos \nu(\pi-\phi,-) \sin \nu(\pi-\phi,-)}{\sin \nu \pi \pm \Delta}.
\]

with

\[
\left(\phi,-\right) = \max_{\phi \in \Phi, \Phi,-} \min(\phi, \Phi). \tag{32}
\]

Similarly, the interior transform function is

\[
f(\nu,\alpha) = (1 + \Gamma) \frac{\cos \nu(\pi-\Phi) \cos \nu(\pi-2\alpha)}{\sin \nu \pi \pm \Delta \sin \nu(\pi-2\alpha)} \tag{33}
\]

According to (31), the symmetric and antisymmetric exterior fields are simply related by a change in sign of \(\Gamma\) and in terms that involve the sum \(\Phi+\phi\).

Since \(|\Gamma|<1\), the poles of (31) and (33) are such that

\[
\nu_n = n + \gamma_n, \quad |\gamma_n| < \frac{1}{\pi} \arcsin|\Gamma| \quad (n \geq 1). \tag{34}
\]

If the wedge angle \(\alpha\) is a rational multiple of \(\pi\), the sequence \(\gamma_n\) becomes periodic and the residue series can be arranged as a finite number of summable series.

III. WEDGE ANGLE A RATIONAL MULTIPLE OF \(\pi\)

When the wedge angle \(2\alpha\) is a rational multiple of \(\pi\), the poles of the above transforms, that is the roots of the denominator function

\[
\sin n \pi \pm \Delta \sin n(\pi-2\alpha) = 0,
\]

are explicitly procurable in terms of the zeros of a polynomial. The normalized supplementary angle

\[
\frac{\pi-2\alpha}{\pi} = \frac{N}{M}
\]

is now expressed in terms of integers such that \(M > N\). With \(\theta = n \pi / M\) the desired roots satisfy

\[
\sin M \theta = \mp \Delta \sin N \theta.
\]

Thus, either \(\sin \theta = 0\) or, with the introduction of \(x = \cos \theta\),

\[
U_{M-1}(x) = \mp \Delta U_{N-1}(x)
\]

in terms of Chebyshev polynomials of the second kind. The first class of roots from \(\theta = k \pi\) are designated

\[
\nu_{kM} = kM \left\{ \begin{array}{ll}
(k \geq 0) & \text{symmetric} \\
(k \geq 1) & \text{antisymmetric}
\end{array} \right. \tag{39}
\]

and the second class of \(M-1\) roots \(x_i^s\) or \(x_i^a\) of (38), labeled in ascending order \((i = 1, 2, \ldots, M-1)\), translate to \(\theta\) values

\[
\theta = [k - \frac{1}{2} + \frac{1}{2} (1)^i] \pi - (1)^i \cos^{-1} x_i
\]

\(i\) \(\equiv\) even

\[
(k \geq 1, 1 \leq i \leq M-1). \tag{40}
\]

In the required transform variable \(\nu\), these roots are denoted

\[
\nu_{(k-1)M+i} = (k - \frac{1}{2}) M + (-1)^i (M/\pi) \sin^{-1} x_i
\]

\(k \geq 1, 1 \leq i \leq M-1). \tag{41}

The fraction \(N/M\) is in its lowest form; if either \(M\) or \(N\) is even, then the antisymmetric \(x\) values are the negatives of the symmetric \(x\) values, i.e., \(x_i^a = -x_{M-i}^s\) \((1 \leq i \leq M-1)\). If \(M\) and \(N\) are both odd, then (38) is a polynomial of degree \((M-1)/2\) in \(x^2\), i.e., \(x_i^s = -x_{M-i}^s\) and \(x_i^a = -x_{M-i}^a\) \([1 \leq i \leq (M-1)/2]\). So the number of independent \(x_i\) values required for both symmetric and antisymmetric solutions is always \((M-1)\).
The residue of the pertinent first term of (31) at \( \nu_{kM} = kM \) is

\[
\frac{(-1)^{k(M-N)} \mp \Gamma}{\pi \sin[(1)^{k(M-N)} \pm N\Gamma/M]}
\cos kM(\pi - \Phi) \sin kM(\pi - \phi),
\]

(42)

while that at \( \nu_{(k-1)M+i} \), defined by (41), is given in terms of Chebyshev polynomials of the first kind by

\[
\frac{T_M(x_i) \pm \Gamma T_N(x_i)}{\pi \sin[T_M(x_i) \pm N\Gamma T_N(x_i)/M]}
\cos \left( \nu_{(k-1)M+i}(\pi - \Phi) \right) \sin \left( \nu_{(k-1)M+i}(\pi - \phi) \right)
\]

\( i \geq N \geq 0, \quad 0 \leq k \leq M - 1 \).

The Chebyshev recursion relation

\[
T_M(x) = U_M(x) - xU_{M-1}(x) = xU_{M-1}(x) - U_M(x) \quad (M \geq 2)
\]

is also true for \( M = 1 \) by interpreting \( U_{-1}(x) \) as zero, which is consistent with \( U_{M-1}(x) = \sin M\theta/\sin \theta \). Subject to (38), the degree of the rational coefficient in (43) is therefore lowered by two in the identity

\[
\frac{T_M(x_i) \pm \Gamma T_N(x_i)}{\pi \sin[T_M(x_i) \pm N\Gamma T_N(x_i)/M]} = \frac{U_{M-2}(x_i) \pm \Gamma U_{N-2}(x_i)}{U_{M-2}(x_i) \pm (N/M)\Gamma U_{N-2}(x_i) \pm (1 - (N/M))\Gamma x_i U_{N-1}(x_i)).
\]

IV. M AND N BOTH ODD

When \( M = 2m + 1, N = 2q + 1 (0 \leq q < m) \), the symmetric and antisymmetric residues can be treated separately and a reduction in the algebra is achieved. In either case, the eigenvalue equation (38) becomes a polynomial of degree \( m \) in \( x_i^2 \) whose roots are structured

\[
x_i^1 = -\beta_m, \quad x_i^2 = -\beta_{m-1}, \ldots, \quad x_i^m = -\beta_1, \quad x_i^{m+1} = \beta_1, \ldots, \quad x_i^{2m} = \beta_m,
\]

(46)
i.e., \( x_{m+1-i} = -\beta_i, x_{m+i} = \beta_i (1 \leq i \leq m) \). The obvious pairings of residues can be exploited by defining \( y = \cos 2\theta = 2x_i^2 - 1 \), whence the eigenvalue equation can be rewritten as

\[
U_m(y) + U_{m-1}(y) = \pm \Gamma U_{q}(y) + U_{q-1}(y),
\]

(47)

where \( U_{-1} = 0, \quad U_0 = 1, \quad U_1 = 2y, \) etc. The eigenvalues are now \( y_i = 2\beta_i^2 - 1 \), i.e.,

\[
\sin^{-1} \beta_i = \frac{1}{\pi} \cos^{-1}(1 - 2\beta_i^2) = \frac{1}{\pi} \cos^{-1}(-y_i).
\]

(48)

Evidently, for any angle \( \psi_i \), (48) and the definition (41) permit the economical summation of the pair of terms

\[
\sum_{k=1}^{\infty} \cos(\nu_{kM+1-i}\psi_i) + \cos(\nu_{(k-1)M+i}\psi_i)
\]

\[= 2 \cos \left( \frac{M\psi}{\pi} \sin^{-1} \beta_i \right) \sum_{k=1}^{\infty} \cos \left( k - \frac{1}{2} \right) M\psi \]

\[= \pi \cos \left( \frac{M\psi}{\pi} \sin^{-1} \beta_i \right) \sum_{n=-\infty}^{\infty} (-1)^n \delta(M\psi/2 - n\pi)
\]

\[= \frac{2\pi}{M} \sum_{n=-\infty}^{\infty} \delta(\psi - (2n\pi/M)T_N(y_i)).
\]

(49)

When the contributions of type (42) are included, the sum of all the symmetric/antisymmetric residues of \( \cos \nu \eta f^{(s)} \) is

\[
\frac{1}{2} \sum_{n=-\infty}^{\infty} \left[ \delta(\eta + |\Phi - \phi| - 2n\pi/M) + \delta(\eta - |\Phi - \phi| - 2n\pi/M) \right]
\]

\[= 2 \sum_{n=-\infty}^{\infty} \frac{T_M(\beta_i) \pm \Gamma T_N(\beta_i)}{MT_M(\beta_i) \pm N\Gamma T_N(\beta_i)} T_N(y_i).
\]

(50)

in which the coefficients in the \( i \) summation can be simplified according to (45). Evidently this summation depends on \( n \), which is restricted to values, dependent on \( N, M \), that allow the vanishing arguments of the four delta functions in (50) to produce field contributions for \( 0 < \eta < \pi \) and the actual range of field and source angles

\[
(m-q)\pi/M < \phi, \quad \Phi < \pi
\]

\[\Rightarrow \begin{cases} (m+1+q)\pi/M < \Phi - \phi < (m+1+q)\pi/M \\ 2(m-q)\pi/M < \phi < \Phi + \Phi < 2\pi \end{cases}.
\]

(51)

Thus the ranges of integer values of \( n \) for each delta function argument are:

\[
\eta + |\Phi - \phi| < 2n\pi/M \quad [1 \leq n \leq (m+q+M)/2],
\]

\[
\eta - |\Phi - \phi| < 2n\pi/M \quad [- (m+q)/2 \leq n \leq m],
\]

\[
\eta + 2\pi - \Phi - \phi < 2n\pi/M \quad (1 \leq n \leq M+q),
\]

\[
\eta - 2\pi + \Phi + \phi < 2n\pi/M \quad [- (m+q) \leq n \leq m].
\]

(52)

The sum of the symmetric/antisymmetric residues of \( e^{-\nu \phi} \sin \nu \pi f^{(s)} \) can be similarly expressed, after the introduction of

\[
\psi_1 = \pi + \Phi - \phi, \quad \psi_2 = \pi - \Phi + \phi, \quad \psi_3 = 3\pi - \Phi - \phi
\]

\[
\psi_4 = \Phi + \phi - \pi, \quad \chi_i = \cos^{-1}(-y_i),
\]

(53)

as
Hence the sum of all the symmetric plus antisymmetric residues of \( \cosh \th ~\cos \phi \) reduces to

\[
1 \sum_{j=1}^{M} \frac{1}{4\pi} \cosh \th ~\cos \phi_j
\]

\[
\times \sum_{i=1}^{m} \frac{T_M(\beta_i) \cos \Gamma T_N(\beta_i)}{T_M(\beta_i) \cos \Gamma T_N(\beta_i)/M}
\]

\[
\times \{ \cosh \left[ \frac{1}{2}(1 + \chi_t / \pi) \xi \right] \sin \left[ \frac{1}{2}(1 - \chi_t / \pi) \phi_t \right]
\]

\[
+ \cosh \left[ \frac{1}{2}(1 - \chi_t / \pi) \xi \right] \sin \left[ \frac{1}{2}(1 + \chi_t / \pi) \phi_t \right] \}
\]  

(54)

in which, since \( M \) is odd,

\[
\frac{1}{2\pi} \sum_{k=0}^{\infty} \frac{(-1)^k + \Gamma}{(-1)^k + \Gamma / M} \left[ \cos kM(\Phi - \phi) + \cos kM(2 \pi - \Phi - \phi) \right] \cos kM \eta
\]

\[
+ \frac{1}{2\pi} \sum_{k=1}^{\infty} \frac{(-1)^k - \Gamma}{(-1)^k - \Gamma / M} \left[ \cos kM(\Phi - \phi) - \cos kM(2 \pi - \Phi - \phi) \right] \cos kM \eta
\]

\[
= \frac{1}{2\pi} \left( \frac{1 - N\Gamma^2/M}{1 - N\Gamma^2/M} \right) \sum_{k=0}^{\infty} \left[ \cos kM(\Phi - \phi + \eta) + \cos kM(\Phi - \phi - \eta) \right]
\]

\[
+ \frac{1}{2\pi} \left( \frac{1 - N\Gamma^2/M}{1 - N\Gamma^2/M} \right) \sum_{n=-\infty}^{\infty} \left[ \delta( \eta + |\Phi - \phi| - 2n \pi/M) + \delta( \eta - |\Phi - \phi| - 2n \pi/M) \right]
\]

\[
= \frac{1}{2\pi} \left( \frac{1 - N\Gamma^2/M}{1 - N\Gamma^2/M} \right) \sum_{n=-\infty}^{\infty} \left[ \delta( \eta + 2\pi - \Phi - \phi - \frac{2n-1}{M} \pi) + \delta( \eta - 2\pi + \Phi + \phi - \frac{2n-1}{M} \pi) \right].
\]  

(56)

For the difference, that is the symmetric minus the antisymmetric residues, interchange the coefficients of these two series. The sum of the symmetric and antisymmetric residues of \( \nu \eta \) \( f(\nu) \) at \( \nu_{(k-1),M+i} \) is

\[
\frac{1}{2\pi} \sum_{k=1}^{\infty} \sum_{i=1}^{M-1} \left[ \frac{T_M(x_t^k) + \Gamma T_N(x_t^k)}{T_M(x_t^k) + N\Gamma T_N(x_t^k)/M} \cos[\nu_{(k-1),M+i}(\pi - \Phi)] \cos[\nu_{(k-1),M+i}(\pi - \phi)] \right]
\]

\[
+ \frac{T_M(x_t^k) - \Gamma T_N(x_t^k)}{T_M(x_t^k) - N\Gamma T_N(x_t^k)/M} \sin[\nu_{(k-1),M+i}(\pi - \Phi)] \sin[\nu_{(k-1),M+i}(\pi - \phi)] \right],
\]  

(57)

which, in view of the relationships

\[
x_t^k = -x_t^{M-i} \quad (1 \leq i \leq M - 1),
\]

\[
\nu_{(k-1),M+i}^{\nu} = (k - \frac{1}{2}) M - (1)^k (M/\pi) \sin^{-1} x_t^{M-i},
\]  

(58)

reduces to

\[
\frac{1}{2\pi} \sum_{k=1}^{M-1} \sum_{i=1}^{M-1} \left[ \frac{T_M(x_t^k) + \Gamma T_N(x_t^k)}{T_M(x_t^k) + N\Gamma T_N(x_t^k)/M} \sum_{n=-\infty}^{\infty} \left[ \delta( \eta + |\Phi - \phi| - \frac{2n\pi}{M}) + \delta( \eta - |\Phi - \phi| - \frac{2n\pi}{M}) \right] T_{2\theta}(x_t^k)
\]

\[
+ \left[ \delta( \eta + 2\pi - \Phi - \phi - \frac{2n-1}{M} \pi) - \delta( \eta - 2\pi + \Phi + \phi - \frac{2n-1}{M} \pi) \right] T_{2\theta-1}(x_t^k) \right].
\]  

(59)

Hence the sum of all the symmetric plus antisymmetric residues of \( \nu \eta \) \( f(\nu) \) is
\[
\frac{1}{2M} \sum_{n=-\infty}^{\infty} \left[ \delta \left( \eta + |\Phi - \phi| - \frac{2n}{M} \pi \right) + \delta \left( \eta - |\Phi - \phi| - \frac{2n}{M} \pi \right) \right] \left[ \frac{1 - N/\sqrt{M}}{1 - N^2/\sqrt{M}^2} + \sum_{i=1}^{M-1} \frac{T_M(x_i^+)}{T_M(x_i^+)} + \frac{N/\sqrt{M}}{N^2/\sqrt{M}^2} \frac{T_N(x_i^+)}{T_N(x_i^+)} \right] T_2\eta(x_i^+)^N \right]
\]

\[
+ \left[ \delta \left( \eta + 2\pi - \Phi - \phi - \frac{2n-1}{M} \pi \right) + \delta \left( \eta - 2\pi + \Phi + \phi - \frac{2n-1}{M} \pi \right) \right] \left[ \frac{1 - N/\sqrt{M}}{1 - N^2/\sqrt{M}^2} + \sum_{i=1}^{M-1} \frac{T_M(x_i^-)}{T_M(x_i^-)} + \frac{N/\sqrt{M}}{N^2/\sqrt{M}^2} \frac{T_N(x_i^-)}{T_N(x_i^-)} \right] T_2\eta(x_i^-)^N \right].
\] 

(60)

For the difference of all symmetric minus antisymmetric residues, interchange the angles $|\Phi - \phi|$ and $2\pi - \Phi - \phi$. The restrictions on the values of $n$ are similar to (52).

The sum of symmetric plus antisymmetric residues of $e^{-x\xi} \sin \nu \pi f^{(c)}$ is

\[
\frac{1}{4\pi} \sum_{k=1}^{\infty} \sum_{i=1}^{M-1} \frac{T_M(x_i^+)}{T_M(x_i^+)} + \frac{N}{\sqrt{M}} \frac{T_N(x_i^+)}{T_N(x_i^+)} e^{-(k-1/2)M \xi} \left[ e^{-(1/2)(M+\pi)\sin^{-1} x_i^+} \sin \left( \left( k - \frac{1}{2} \right) M - (1) \frac{M}{\pi} \sin^{-1} x_i^+ \right) (\pi + |\Phi - \phi|) \right]
\]

\[
+ \sin \left( \left( k - \frac{1}{2} \right) M - (1) \frac{M}{\pi} \sin^{-1} x_i^+ \right) (\pi - |\Phi - \phi|) \right] + 2 \text{ similar terms with } 2\pi - \Phi - \phi \text{ instead of } |\Phi - \phi| \right] + e^{(1/2)(M+\pi)\sin^{-1} x_i^+} \sin \left( \left( k - \frac{1}{2} \right) M - (1) \frac{M}{\pi} \sin^{-1} x_i^+ \right) (\pi + |\Phi - \phi|) \right]
\]

\[
+ \sin \left( \left( k - \frac{1}{2} \right) M - (1) \frac{M}{\pi} \sin^{-1} x_i^+ \right) (\pi - |\Phi - \phi|) \right]
\]

\[- 2 \text{ similar terms with } 2\pi - \Phi - \phi \text{ instead of } |\Phi - \phi| \right].
\]

(61)

For any angle $\psi$ and $|\gamma| \leq 1/2$, the infinite series above are summable as

\[
\sum_{i=1}^{\infty} \left[ \sin \left( \left( k - \frac{1}{2} \right) + (1) \frac{\gamma}{M} \right) \right] \psi \exp \left\{ \left[ - (k - \frac{1}{2}) + (1) \frac{\gamma}{M} \right] \right\} \exp \left\{ \left[ (k - \frac{1}{2}) - (1) \frac{\gamma}{M} \right] \right\}
\]

\[
= 2 \Im \sum_{k=1}^{\infty} e^{-(1/2)M \xi - i(1)\frac{\gamma}{M} \sinh \left( \gamma M (\xi - i \psi) \right)}
\]

\[
= \Im \left[ \frac{\cosh \left( \gamma M (\xi - i \psi) \right)}{\sinh \left( \gamma M (\xi - i \psi) \right)} \right]
\]

\[
= \frac{\cosh \left( M \ξ (1/2 + \gamma) \right) \sin \left( M \psi (1/2 - \gamma) \right) \pm \cosh \left( M \xi (1/2 - \gamma) \right) \sin \left( M \psi (1/2 + \gamma) \right)}{\cosh M \xi \mp \cos M \psi}
\]

(62)

and so the sum of the symmetric plus antisymmetric residues of $e^{-x\xi} \sin \nu \pi f^{(c)}$ is the finite sum

\[
\frac{1}{4\pi} \sum_{i=1}^{M-1} \frac{T_M(x_i^+)}{T_M(x_i^+)} + \frac{N}{\sqrt{M}} \frac{T_N(x_i^+)}{T_N(x_i^+)} \left[ \sum_{j=1}^{2} \cosh \left( M \xi (1/2 + \lambda_j) \right) \sin \left( M \psi_j (1/2 - \lambda_j) \right) \right] + \cosh \left( M \xi (1/2 - \lambda_j) \right) \sin \left( M \psi_j (1/2 + \lambda_j) \right)
\]

(63)

\[
+ \sum_{j=3}^{4} \cosh \left( M \xi (1/2 + \lambda_j) \right) \sin \left( M \psi_j (1/2 - \lambda_j) \right) - \cosh \left( M \xi (1/2 - \lambda_j) \right) \sin \left( M \psi_j (1/2 + \lambda_j) \right)
\]

with $\psi_j \ (j = 1, 2, 3, 4)$ defined by (53) and $\lambda_j = (1/n) \sin^{-1} x_i^+$. An interchange of the $j$ values yields the difference of the symmetric minus the antisymmetric residues.
VI. THE TOTAL FIELD AND SOME SPECIAL CASES

The results (50), (54), (60), (63) now enable the total field to be recovered by substituting these residue sums into (21), bearing in mind that, as stated in Sec. II, \( p_s + p_a \) is the solution when the source angle \( \Phi \) and the observation angle \( \phi \) are both on the same side of the wedge bisector, while \( p_s - p_a \) applies when \( \Phi \) and \( \phi \) are separated by the symmetry plane. The possible occurrence of delta functions of the general form \( \delta(\eta-\psi) \) where \( \eta \) is defined in (22), is at the time instant

\[
t_0 = \frac{1}{c} (r^2 + R^2 + z^2 - 2rR \cos \psi)^{1/2}.
\]  

(64)

The scaling relation

\[
\frac{c}{rR} \sin \eta \delta(\eta-\psi) = \delta(t-t_0) dt
\]

(65)

allows these delta functions to be written in the expected form

\[
\frac{c}{rR} \sin \eta \delta(\eta-\psi) = \frac{\delta(t-(1/c)(r^2 + R^2 + z^2 - 2rR \cos \psi)^{1/2})}{(r^2 + R^2 + z^2 - 2rR \cos \psi)^{1/2}}.
\]

(66)

Before proceeding to the special cases, note first that if \( N=1 \), the coefficient in (45) reduces to

\[
\frac{U_{M-2}(x_i)}{U_{M-2}(x_i) \pm \left[ 1 - \frac{1}{M} \right] \Gamma x_i}
\]

(67)

which in turn, if \( M=2m+1 \), can be written as

\[
\frac{U_{m-1}(y_i)}{U_{m-1}(y_i) \pm m \Gamma/M}.
\]

(68)

A. Case \( \alpha = \pi/3 \)

In this case \( M=3 \), \( N=1 \) and only the values of \( \pm n \) up to 3 are relevant in (50). The eigenvalue equation (38) has the single solution

\[
y_1 = -\frac{2}{3}(1 \pm \Gamma)
\]

(69)

and the coefficient in (50) is now

\[
\frac{1 \pm \Gamma}{2(3 \pm \Gamma)} + \frac{U_0(y_1)}{3U_0(y_1) \pm \Gamma} T_n(y_1)
\]

\[
= \frac{1}{3 \pm \Gamma} \left[ 1 \pm \frac{1}{2} + T_n \left( \frac{1 \pm \Gamma}{2} \right) \right].
\]

(70)

This is simplified by the observation that \( T_n(-x) + x \) has the factor \( 1 + x \) to yield, for \( n = 0,1,2,3 \), respectively:

\[
1/2, 0, \pm \Gamma/2, \text{ and } (1-\Gamma^2)/2.
\]

(71)

The diffracted field expression (54) is simplified with

\[
\chi = \cos^{-1}\left[ \frac{1}{2}(1 \pm \Gamma) \right] \quad (0 < \chi < \pi/2), \quad \text{i.e.,}
\]

\[
\sin \frac{2}{3}(\pi \mp \chi) = -\cos \frac{2}{3}\Gamma \sqrt{3 \pm \Gamma}
\]

and substitution in (21) yields

\[
p^{(e)} = -\frac{c}{8 \pi rR} \sin \eta \left[ \delta(\eta - |\Phi - \phi|) + \delta(\eta - 2 \pi + \Phi + \phi) \pm \Gamma \delta(\eta - 4 \pi/3 + |\Phi - \phi|) \right]
\]

\[
\pm \Gamma \delta(\eta + 2 \pi/3 - \Phi - \phi) + (1 - \Gamma^2) \delta(\eta - \Phi - \phi) \right] + \frac{3 \Gamma}{8 \pi^2 (3 \pm \Gamma)^{1/2} rR} \sinh \xi \left[ \cosh 3 \xi + \cos 3(\Phi - \phi) \right]
\]

\[
\times \left[ \cosh\left[ \frac{x}{2} (1 + \chi/\pi) \xi \right] \cosh\left[ \frac{x}{2} (1 - \chi/\pi) (\Phi - \phi) \right] + \sinh\left[ \frac{x}{2} (1 - \chi/\pi) \xi \right] \sinh\left[ \frac{x}{2} (1 + \chi/\pi) (\Phi - \phi) \right] \right]
\]

\[
\pm \cosh 3 \xi + \cos 3(\Phi + \phi) \left[ \cosh\left[ \frac{x}{2} (1 + \chi/\pi) \xi \right] \cos\left[ \frac{x}{2} (1 - \chi/\pi) (2 \pi - \Phi - \phi) \right] \right.
\]

\[
+ \cosh\left[ \frac{x}{2} (1 - \chi/\pi) \xi \right] \cos\left[ \frac{x}{2} (1 + \chi/\pi) (2 \pi - \Phi - \phi) \right] \right].
\]

(73)

The \( \delta \) functions have range restrictions and, after using (52), the geometrical acoustics field (delta function contributions) caused by a point source on the same side of the symmetry plane \( (p_s^{(e)} + p^{(e)}_a) \) is therefore

\[
p^{(e)} = -\frac{1}{4 \pi} \left[ \delta[t - (1/c)(r^2 + R^2 + z^2 - 2rR \cos(\phi - \Phi))]^{1/2} \right]
\]

\[
+ \Gamma \delta[t - (1/c)(r^2 + R^2 + z^2 - 2rR \cos(\phi + \Phi - 2 \pi/3))]^{1/2} \right] H(5 \pi/3 - \phi - \Phi) \right].
\]

(74)
while, if the source angle $\Phi$ is measured in the opposite sense as the field angle $\phi$ (Fig. 4), then $(p_s^{(e)} - p_a^{(e)})$ leaves

$$p_s^{(e)} = \frac{1}{4\pi} \left[ \delta(t-(1/c)[r^2+R^2+z^2-2rR \cos(2\pi-\phi-\Phi)])^{1/2} H(\phi+\Phi-\pi) + \delta(t-(1/c)[r^2+R^2+z^2-2rR \cos(4\pi/3-\phi-\Phi)])^{1/2} H(\phi+\Phi-\pi/3) + (1-\Gamma^2) \delta(t-(1/c)[r^2+R^2+z^2-2rR \cos(\phi+\Phi)])^{1/2} H(\pi-\phi-\Phi) \right].$$

(75)

The first term in each of (74) and (75) is the direct ensonification from the point source, while the second terms are reflections from either wedge face. Penetration through the wedge is responsible for the last term of (75), where the total transmission coefficient is $(1+\Gamma)(1-\Gamma)$.

**B. Case $\alpha=2\pi/5$**

With $M=5$, $N=1$, the applicable range of $\pm n$ is $0$ to $5$ and the eigenvalue equation (47) is

$$U_1(y) + U_2(y) = 2y + 4y^2 - 1 = \mp \Gamma,$$

i.e.,

$$y_1 + y_2 = -\frac{1}{2}, \quad y_1 y_2 = -\frac{1}{4}(1 \mp \Gamma),$$

(77)

The coefficient in (50) is thus

$$\frac{1 \pm \Gamma}{2(5 \pm \Gamma)} + \sum_{i=1}^{2} \frac{U_1(y_i)}{5 U_1(y_i)} T_n(y_i) = \frac{1}{5 \pm \Gamma} \left[ \frac{1 \pm \Gamma}{2} + T_n(y_2) + T_n(y_1) \right]$$

$$+ \frac{1}{2} \left[ \Gamma T_n(y_2) - T_n(y_1) \right] \frac{y_2-y_1}{y_2-y_1},$$

(78)

which produces, for $n = 0, 1, 2, ..., 5$, respectively,
where, from (76) and (81)
\[
\chi_1 \pm \chi_2 = \cos^{-1}\{ - \frac{1}{4} [1 + \Gamma \pm \sqrt{(5 + \Gamma)(1 + \Gamma)}] \}. \tag{83}
\]

Results corresponding to (74) and (75) can readily be deduced for the antisymmetric field by changing the sign of \(\Gamma\) and the sign of terms involving the sum \(\Phi + \phi\) in this symmetric field expression.

**C. Case \(\alpha = \pi/4\)**

In this case of the right-angled wedge, \(N/M = 1/2\) and the eigenvalue equations (39), (41) reduce to
\[
\nu_{2k} = 2k, \quad \nu_{2k-1}^a = (2k-1) + (-1)^k (2/\pi) \sin^{-1}(\Gamma/2). \tag{84}
\]

The residue sum (60) reduces to
\[
\frac{1}{4-\Gamma^2} \sum_{n=-\infty}^{\infty} \left\{ [\delta(\eta + |\Phi - \phi| - n \pi) + \delta(\eta - |\Phi - \phi| - n \pi)] [T_{2n}(\Gamma/2) - T_{2}(\Gamma/2)] + \left[ \delta(\eta + 2\pi - \Phi - \phi - \frac{2n-1}{2} \pi) + \delta(\eta - 2\pi + \Phi + \phi - \frac{2n-1}{2} \pi) \right] \left[ \Gamma/2 - T_{2n-1}(\Gamma/2) \right] \right\}
\] (85)
in which the relevant values of \(n\) are (1), (0), (1), (0, 1) in the respective delta functions above. Since the only nonzero contributions arise from
\[
T_0(\Gamma/2) - T_2(\Gamma/2) = \frac{1}{4}(4 - \Gamma^2), \quad \Gamma/2 - T_3(\Gamma/2) = \frac{1}{4}\Gamma(4 - \Gamma^2)
\] (86)
the total sum reduces to
\[
\frac{1}{4} \delta(\eta - |\Phi - \phi|) + \Gamma \delta(\eta + \pi/2 - \Phi - \phi). \tag{87}
\]

After substituting (63) and (87) into (21), the field due to a point source on the same side of the symmetry plane is therefore
\[
p_s^{(e)} + p_a^{(e)} = \frac{-c}{4\pi R \sin \eta} \left[ \delta(\eta - |\Phi - \phi|) + \Gamma \delta(\eta + \pi/2 - \Phi - \phi) \right]
\]
\[
+ \frac{e \Gamma}{2\pi^2 R \sinh \xi(4-\Gamma^2)^{1/2}} \left[ \frac{1}{\cosh 2\xi - \cos 2(\Phi + \phi)} \right] \left[ \cos \left( 1 + \frac{2}{\pi} \sin^{-1}(\Gamma/2) \right) \xi \cos \left( 1 - \frac{2}{\pi} \sin^{-1}(\Gamma/2) \right) \right]
\]
\[
\times \left( \Phi - \phi \right) - \cosh \left( 1 - \frac{2}{\pi} \sin^{-1}(\Gamma/2) \right) \xi \cos \left( 1 + \frac{2}{\pi} \sin^{-1}(\Gamma/2) \right) \left( \Phi - \phi \right) \tag{88}
\]
\[
\times \left[ \cosh \left( 1 + \frac{2}{\pi} \sin^{-1}(\Gamma/2) \right) \xi \cos \left( 1 - \frac{2}{\pi} \sin^{-1}(\Gamma/2) \right) \left( 2\pi - \Phi - \phi \right) \right]
\]
\[
+ \cosh \left( 1 - \frac{2}{\pi} \sin^{-1}(\Gamma/2) \right) \xi \cos \left( 1 + \frac{2}{\pi} \sin^{-1}(\Gamma/2) \right) \left( 2\pi - \Phi - \phi \right) \right] .
\]

By a similar sequence of arguments, the field due to a point source on the opposite side of the symmetry plane is
\[
p_s^{(e)} - p_a^{(e)} = \frac{-c}{4\pi R \sin \eta} \left[ \delta(\eta - 2\pi + \Phi + \phi) + \Gamma \delta(\eta + |\Phi - \phi| - 3\pi/2) + (1 - \Gamma^2) \delta(\eta - \Phi - \phi) \right]
\]
\[
+ \frac{e \Gamma}{2\pi^2 R \sinh \xi(4-\Gamma^2)^{1/2}} \left[ \frac{1}{\cosh 2\xi + \cos 2(\Phi - \phi)} \right] \left[ \cos \left( 1 + \frac{2}{\pi} \sin^{-1}(\Gamma/2) \right) \xi \cos \left( 1 - \frac{2}{\pi} \sin^{-1}(\Gamma/2) \right) \right]
\]
\[
\times \left( 2\pi - \Phi - \phi \right) - \cosh \left( 1 + \frac{2}{\pi} \sin^{-1}(\Gamma/2) \right) \xi \cos \left( 1 + \frac{2}{\pi} \sin^{-1}(\Gamma/2) \right) \left( 2\pi - \Phi - \phi \right) \right]
\]
\[
- \frac{1}{\cosh 2\xi + \cos 2(\Phi - \phi)} \left[ \cosh \left( 1 + \frac{2}{\pi} \sin^{-1}(\Gamma/2) \right) \xi \cos \left( 1 - \frac{2}{\pi} \sin^{-1}(\Gamma/2) \right) \left( \Phi - \phi \right) \right]
\]
\[
+ \cosh \left( 1 + \frac{2}{\pi} \sin^{-1}(\Gamma/2) \right) \xi \cos \left( 1 + \frac{2}{\pi} \sin^{-1}(\Gamma/2) \right) \left( \Phi - \phi \right) \right] . \tag{89}
\]
When \( N/M = 2/3 \) the eigenvalue equation (38) yields
\[
\begin{align*}
    x_1^1 &= -\frac{1}{2} \Gamma - \frac{3}{4} (1 + \frac{3}{4} \Gamma^2)^{1/2} < 0; & x_2^1 &= -\frac{1}{2} \Gamma + \frac{3}{4} (1 + \frac{3}{4} \Gamma^2)^{1/2} > 0 \\
    x_1^2 &= \frac{1}{2} \Gamma - \frac{3}{4} (1 + \frac{3}{4} \Gamma^2)^{1/2} < 0; & x_2^2 &= \frac{1}{2} \Gamma + \frac{3}{4} (1 + \frac{3}{4} \Gamma^2)^{1/2} > 0
\end{align*}
\]
i.e., from (41),
\[
\nu_{a_{3k-2}} ^{a_{3k-1}} = \frac{3}{2} \left( k - \frac{1}{2} \right) \left( -1 \right)^{\frac{3}{2} - \frac{1}{2} \sin^{-1} x_2^a}, \quad \nu_{a_{3k-2}} ^{a_{3k-1}} = \frac{3}{2} \left( k - \frac{1}{2} \right) \left( -1 \right)^{\frac{3}{2} - \frac{1}{2} \sin^{-1} x_2^a}.
\]
The coefficient in (57) can, by use of (90), be simplified to
\[
\frac{T_3(x_i^1) + \Gamma T_3(x_i^2)}{T_3(x_i^1) + 2 \Gamma T_2(x_i^1)/3} = \frac{3}{9 - 4 \Gamma^2} \left[ \Gamma (1 - \Gamma^2) \right] \frac{3 - 2 \Gamma^2}{9 - 4 \Gamma^2} + \frac{3 - 2 \Gamma^2}{9 - 4 \Gamma^2} \left[ T_{2n}(x_i^1) + T_{2n}(x_i^2) \right]
\]
\[
= \frac{1}{2} \left[ \delta(\eta - |\Phi - \phi| - 2n \pi/3) + \delta(\eta + |\Phi - \phi| - 2n \pi/3) \right] \left[ 3 - 2 \Gamma^2 \right] \left[ 9 - 4 \Gamma^2 \right] + \frac{1}{2} \left[ \delta(\eta + 2 \pi - \Phi - \phi - 2n - 1/3 \pi) + \delta(\eta - 2 \pi + \Phi + \phi - 2n - 1/3 \pi) \right]
\]
\[
\times \left[ (9 - 4 \Gamma^2) + \frac{3 - 2 \Gamma^2}{9 - 4 \Gamma^2} \left[ T_{2n-1}(x_i^1) + T_{2n-1}(x_i^2) \right] \right] \frac{\Gamma (1 - \Gamma^2)}{(9 - 4 \Gamma^2) (4 + \Gamma^2)^{1/2}}
\]
\[
= \frac{1}{2} \left[ \delta(\eta - |\Phi - \phi|) + \Gamma \delta(\eta + 2 \pi - \Phi - \phi) H(3 \pi - 10 \alpha - \Phi - \phi) - \Gamma (1 - \Gamma^2) \delta(\eta - 2 \pi - \Phi - \phi) \right]
\]
\[
\times H(3 \pi - 14 \alpha - \Phi - \phi),
\]
(93)

VII. SAMPLE RESULTS FOR THE DIFFRACTION FIELD

Temporal and spatial behavior of the diffraction field is succinctly displayed as a function of the single metric variable \( \xi \) of Eq. (22). The dominant factor \( \sinh \xi \) in the denominator of the second term of Eq. (21) produces a first-order singularity at the onset \( (\xi = 0) \) and exponential decay (for large \( \xi \)), characteristic of the “afterglow” of a time-domain field whose equivalent source is the infinitely long wedge apex. The suppression of this known factor greatly reduces the dynamic range required of linear scale graphs. Therefore, the variation of the diffraction field \( p^\phi_d(\xi) \) with source location \( (R, \Phi) \), observation point \( (r, \phi, z) \), and density contrast (in terms of \( \Gamma \)) is simplified by examining the scaled diffraction field \( p^\phi_d(\xi) / rR \sinh \xi \bar{c} |\Gamma| \) vs \( \xi \). The additional factor \( rR/c \) is obvious from Eq. (21), while the normalizing factor \( |\Gamma| \) lowers the sensitivity of the resultant graphs to \( \Gamma \). Adoption of this scaled diffraction field eliminates any need to consider individual numerical values for the distances \( R, r, \) and \( z \), as well as the time \( t \). However, the inherent angular dependence of the solution requires the examination of a range of observation angles \( \phi \) for a fixed source coordinate \( \Phi \).

All of the calculations reveal that the diffraction from a penetrable wedge \((-1 < \Gamma < +1)\) is simply related to that of the corresponding hard or soft wedge \((\Gamma = \pm 1)\). Based on the sample wedges considered here, it is apparent that this basic relationship persists over a range of wedge angles. Therefore,
the purposes of the present study are adequately served by graphing, in Figs. 6–10, the parametrized results for one typical case, the corner wedge \( \alpha = \pi/4 \), with \( \alpha = \pi/4 \) and \( \Phi = \pi/2 \), the geometrical acoustics field suffers discontinuities at \( \phi = \pi \) (the reflection boundary) and at \( \phi = 3\pi/2 \) (the shadow boundary). The diffraction field is most important in the vicinity of these ray boundaries, where its primary role is to smooth-out the step discontinuities in the geometrical acoustics field. The second terms of Eqs. (88) and (89) reveal that this discontinuity at the reflection boundary is proportional to \( \Gamma \), while the discontinuity at the shadow boundary is proportional to \( \Gamma^2 \), as indicated by the third term of Eq. (89).

The variation of the scaled diffraction field with \( \xi \) is graphed in Fig. 6 for the subject soft wedge \( \Gamma = -1 \) at 13 observation angles \( (100^\circ \leq \phi \leq 220^\circ) \) that border the reflec-

FIG. 6. Scaled diffraction field \( p_d^{(r)} \times r R \sinh \beta c |\Gamma| \) as a function of the metric variable \( \xi \). Case: \( \Gamma = -1, \alpha = \pi/4, \Phi = \pi/2, 100^\circ \leq \phi \leq 220^\circ \).

FIG. 8. Scaled diffraction field \( p_d^{(r)} \times r R \sinh \beta c |\Gamma| \) as a function of the metric variable \( \xi \). Case: \( \Gamma = -1, \alpha = \pi/4, \Phi = \pi/2, 230^\circ \leq \phi \leq 310^\circ \).

FIG. 7. Scaled diffraction field \( p_d^{(r)} \times r R \sinh \beta c |\Gamma| \) as a function of the metric variable \( \xi \). Case: \( \Gamma = -0.1, \alpha = \pi/4, \Phi = \pi/2, 100^\circ \leq \phi \leq 220^\circ \).

FIG. 9. Scaled diffraction field \( p_d^{(r)} \times r R \sinh \beta c |\Gamma| \) as a function of the metric variable \( \xi \). Case: \( \Gamma = -0.5, \alpha = \pi/4, \Phi = \pi/2, 230^\circ \leq \phi \leq 310^\circ \).

tion boundary at \( \phi = 180^\circ \). Note the change of sign and the higher initial bang \( (\xi \to 0) \) as the reflection boundary is crossed. When the reflection coefficient is changed to \( \Gamma = -0.1 \), the curves of Fig. 7 apply. Recall the use of \( \Gamma \) as a scaling factor. The qualitative similarity of the curves in Figs. 6 and 7 suggests that the presence of penetration does not dramatically alter the general nature of the external diffrac-

The shadow boundary at \( 270^\circ \) is the subject of Figs. 8 and 9, where the observation angle is varied from \( 230^\circ \) to \( 310^\circ \) for \( \Gamma = -1 \) and \( -0.5 \), respectively. The diffraction fields of Fig. 9 are lower in magnitude, as anticipated from the weaker discontinuity in the geometrical acoustics field.
Furthermore, this trend persists as the density contrast is varied over the entire range \(-1 \leq \Gamma \leq +1\).

Another view of the diffraction mechanism is provided by Fig. 10, which illustrates the initial bang of the scaled diffraction field

$$\lim_{\xi \to 0} p_d^{(1)}(\xi) \times rR \sinh \xi |\Gamma|$$

as a function of observation angle \(\phi\). Case: \(\alpha = \pi/4, \Phi = \pi/2, \Gamma = -1, -0.1, +0.1, +1\). The graphs for intermediate values of \(\Gamma\) all fall continuously between the selected curves. Again, note the change in sign at each ray boundary discontinuity and the shift in zero crossings.

\section*{VIII. CONCLUSIONS}

Closed-form expressions for the complete pressure field caused by an impulsive point source in the region exterior to a density-contrast wedge are obtained for several specific wedges. The geometrical acoustics terms are consistent with the appropriate Snell’s law ansatz, including reflection and shadow boundaries, multiple reflections, and the usual Fresnel reflection and transmission coefficients. In the presence of penetration, the diffracted field is structurally similar to the Biot–Tolstoy results, with the primary difference being a lower amplitude attributable to \(\Gamma\) or \(\Gamma^2\). The persistent qualitative behavior is the initial singularity at the arrival time in accordance with the generalized Fermat’s principle, followed by the exponential tail or “afterglow” of a time-domain field whose equivalent source is the infinitely long wedge apex.