Acoustic scattering by a rigid elliptic cylinder in a slightly viscous medium

Robert W. Scharstein
Electrical Engineering Department, University of Alabama, Tuscaloosa, Alabama 35487-0286

Anthony M. J. Davis
Mathematics Department, University of Alabama, Tuscaloosa, Alabama 35487-0350

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A complete solution is obtained for the two-dimensional diffraction of a time-harmonic acoustic plane wave by an impenetrable elliptic cylinder in a viscous fluid. Arbitrary size, ellipticity, and angle of incidence are considered. The linearized equations of viscous flow are used to write down expressions for the dilatation and vorticity in terms of products of radially and angular dependent Mathieu functions. The no-slip condition on the rigid boundary then determines the coefficients. The resulting computations are facilitated by recently developed library routines for complex input parameters. The solution for the circular cylinder serves as a guide and a differently constructed solution for the strip is also given. Typical results in the “resonant” range of dimensionless wave number, displaying the surface vorticity and the far-field scattering pattern are included, with the latter allowing comparison with the inviscid case. © 2007 Acoustical Society of America. [DOI: 10.1121/1.2727332]

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I. INTRODUCTION

The elliptic cylinder is an interesting and important two-dimensional scatterer of finite cross section because it is simple enough to conform to coordinate surfaces where the wave equation is separable, and it naturally includes a degree of geometric flexibility, i.e., the ellipticity. The two limiting cases of the ellipse, namely the strip and the circle, are important canonical scatterers. Although a fundamental study, the physical insight gained from this paper has important implications for engineering systems that exploit the acoustic signatures of elongated bodies in real fluid media: One obvious application is antisubmarine warfare. The effect of a small kinematic viscosity upon the scattered acoustic field is analyzed, with particular attention to the vorticity of the velocity field that exists in a boundary layer close to the surface of the hard elliptic cylinder. The appropriate no-slip boundary condition was first applied by Alblas to include the effect of a small viscosity in the classic Sommerfeld half-plane problem. A sequence of studies of viscosity effects on sound scattering by Davis and Nagem considered the half plane, the circular aperture, and the circular disk, avoiding possible ambiguities associated with the Helmholtz representation by writing down the solution form of the pressure and deducing expressions for the velocity components before applying the boundary conditions. This method tended to merge the distinct contributions from the dilatation (div) and vorticity (curl) but, in a further paper, Davis and Nagem constructed the scattered field due to a solid or elastic sphere by deducing the velocity components directly from the dilatation and vorticity.

Very few previous authors have included viscous effects and very few have given complete solutions in terms of elliptic coordinates. The most notable predecessor of this paper is by Barakat, who used the notation of Morse and Feshbach to construct far field approximations to plane wave inviscid diffraction by an elliptic cylinder. Without modern computational power, the far field was necessarily the focus of attention, yielding tables of results. This paper fills that computational gap and shows how to work from the dilatation and vorticity whose governing equations, but not boundary conditions, are independent.

Murga used an ad hoc combination of potential and boundary layer theory to study the two-dimensional (2D) half plane problem but this method did not display the Stokes wave feature. Tsoi used the Helmholtz representation and a Watson transformation to consider the high frequency far field approximation. Zhuk retained inertia terms to predict the time averaged force on a solid circular cylinder. Hinders studied the scattering by liquid or elastic spheres but with rather lengthy algebra, despite his neglect of the small damping factor in the acoustic component of the external field. Homentkovschi et al. introduced the unnecessary complications of singular integral equations for 2D scattering by a planar array of strips.

Elliptic geometry appears in the inviscid literature but no serious use is made of elliptic coordinates which are often unmentioned. Chinnery and Humphrey, with an impedance boundary condition, give a solution in terms of Mathieu functions but add only experimental results. Leon et al. prefer to handle the multiple interactions in a Fourier modal method and give an extensive reference list.

The basic theory to accommodate a small viscosity is summarized in Sec. II and the technique of simultaneously considering the vorticity and divergence of the acoustic ve-
locity is first explained in context of the simpler circle geometry. The full treatment of the elliptic cylinder is presented in Sec. III. Separation of variables of the relevant Helmholtz equations in elliptic cylinder coordinates introduces the celebrated Mathieu functions. A library of Mathieu function routines developed by Wilson and Scharstein\textsuperscript{16} facilitates the calculations which require complex input parameters. Mathieu functions of both the angular and radial type depend upon a parameter that derives from the wave number in the appropriate Helmholtz equation, as well as the physical dimensions of the ellipse. This parameter is complex-valued because of the presence of viscosity, and in fact the divergence (dilatation) and curl (vorticity) expansions require two different parameters. This feature complicates the analysis and calculations are performed that exploit the structure and make efficient use of the Mathieu function numerics.\textsuperscript{16} Interpretation of the boundary layer nature of the vorticity is aided by a WKB(J) or LG asymptotic expansion of the radial Mathieu functions having a large imaginary parameter.

Expressions for the far scattered field are summarized in Sec. IV. An independent analysis of the limiting case of the strip is the subject of Sec. V, where expansions of the normal and tangential surface stress explicitly include the proper edge-condition singularity dictated by viscous flow. Computational results of the surface vorticity and far scattered field pattern are discussed in Sec. VI, with due attention to the relationship between the ellipse and its degenerate (circle and strip) forms.

II. THEORY AND CIRCULAR CYLINDER

The standard acoustic equations for linearized flow in a homogeneous viscous fluid medium are the equation of continuity

$$\frac{\partial p}{\partial t} + \rho_0 \nabla \cdot \mathbf{v} = 0,$$

the momentum equation

$$\frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{\rho_0} \nabla p + \nu \nabla^2 \mathbf{v} + \frac{\nu}{3} \nabla (\nabla \cdot \mathbf{v}),$$

and the equation of state

$$\frac{dp}{d\rho} = c_0^2,$$

in which $\mathbf{v}$ is the fluid velocity vector, $\rho_0$ is the ambient fluid density, $\rho$ is the density perturbation, $p$ is the fluid pressure, $c_0$ is the sound speed, and $\nu$ is the kinematic viscosity. Equation (2) assumes a Stokesian fluid for which the convective part of the acceleration is neglected. It may be deduced from Eqs. (1)–(3) that the vorticity $\Omega = \nabla \times \mathbf{v}$ satisfies

$$\frac{\partial \Omega}{\partial t} = \nu \nabla^2 \Omega,$$

as in unsteady creeping flow, while $p$ and $E=\nabla \cdot \mathbf{v}$ satisfy an acoustic wave equation with viscous damping, namely

$$\frac{\partial^2}{\partial t^2} [p, E] = \left( \frac{c_0^2}{\rho_0} + \frac{4}{3} \frac{\nu}{\rho_0} \frac{\partial}{\partial t} \right) \nabla^2 [p, E].$$

In the 2D disturbance of period $2\pi/\omega$ considered here, $E=E(x, y)$, $\Omega=2\Omega(x, y)$ and the time-harmonic dependence $\exp(-i\omega t)$ is suppressed. Then Eqs. (4) and (5) reduce to

$$\left( \nabla^2 + k_0^2 \right) [p, E] = 0,$$

$$\left( \nabla^2 + i\omega \overline{v} \right) \Omega = 0,$$

where the complex acoustic wave number is

$$k = \frac{k_0}{\sqrt{1 - 4i\epsilon^2/3}},$$

with

$$k_0 = \omega/c_0, \quad \epsilon^2 = \omega \nu c_0^2 \ll 1.$$

The pressure $p$ and dilatation $E$ are related, according to Eqs. (1) and (3), by

$$\frac{p}{\rho_0 c_0^2} = \frac{E}{\omega}.$$

The geometric configuration for the elliptic cylinder diffraction is illustrated in Fig. 1. The exciting plane wave (irrotational) propagates in the $\phi_0 + \pi$ direction, that is, incident from the azimuthal angle $\phi_0$, and is given by

$$\mathbf{v} = \frac{\nu_0}{k} \nabla \left[ e^{-ik(\cos \phi_0 \cos \phi \sin \phi)} \right].$$

The scattered field $\mathbf{v}$ is determined by application of the no-slip condition $\mathbf{v} + \mathbf{v}' = 0$ at the ellipse. Two direct methods of solution, avoiding the introduction of potentials, are available for the ellipse scattering. One involves writing down a solution form for the pressure and using the momentum equation to establish associated forms for the velocity components. Alternatively, the latter can be deduced from solution forms for the dilatation $E$ and vorticity $\Omega$. The second method is adopted here except for the limiting case of the strip. To facilitate understanding of the ellipse analysis; the simpler circle case is presented first.

If $b \to a$ the resulting symmetrical scatterer does not require any incidence angles other than $\phi_0=0$. The incident plane wave (irrotational) propagating in the $-x$ direction is

$$\mathbf{v}_i = \frac{\nu_0}{k} \nabla \left[ e^{-ikx} \right],$$

in which

![FIG. 1. Plane wave incident upon elliptic cylinder.](image-url)
dependence of the vorticity on both plane. Suitable solutions of Eq. (5) are

\[ v^i = v_0 \left[ J_0(kr) + 2 \sum_{n=1}^{\infty} i^n J_n(kr) \cos n\phi \right] \]

the incident wave components are given by

\[ v^i = v_0 \left[ J_0(kr) + 2 \sum_{n=1}^{\infty} i^n J_n(kr) \cos n\phi \right] \]

In the inviscid case, only the normal component of the total velocity must vanish at the circle. The vanishing of the \( n = 0 \) term in Eq. (14) shows that scattering of the axisymmetric component is unaffected by the presence of viscosity. The \( n \neq 0 \) Fourier components in Eq. (14) generate viscosity-driven vorticity analogous to the Stokes wave in a viscous fluid bounded by a tangentially vibrating plane. However, dependence of the vorticity on both \( r \) and \( \phi \) means that the acoustic diffraction has to be modified, as in reflection at a plane. Suitable solutions of Eq. (6), constructed to display the above-described physics, are

\[ E(r, \phi) = v_0 k \left[ \frac{H_0^{(1)}(kr)}{H_0^{(1)}(ka)} + 2 \sum_{n=1}^{\infty} i^n \frac{H_n^{(1)}(kr)}{H_n^{(1)}(ka)} \cos n\phi \right] \]

\[ \Omega(r, \phi) = 2v_0 \kappa^2 \sum_{n=1}^{\infty} i^n \frac{B_n H_n^{(1)}(kr)}{n H_n^{(1)}(ka)} \sin n\phi, \]

where

\[ \kappa = \sqrt{\frac{i \omega}{v}} = \frac{k_0}{\epsilon} \frac{1 + i}{\sqrt{2}}. \]

The equations

\[ \frac{\partial}{\partial r} (rv_r) + \frac{\partial}{\partial \phi} v_\phi = Er \]

\[ \frac{\partial}{\partial r} (rv_\phi) - \frac{\partial}{\partial \phi} v_r = \Omega r \]

are then solved by observing that Bessel’s Equation implies that

\[ rH_n^{(1)}(kr)e^{i\phi} = \frac{\partial}{\partial r} \left[ -\frac{r}{k} H_n^{(1)}(kr)e^{i\phi} \right] + \frac{\partial}{\partial \phi} \left[ -\frac{i n}{k^2 r} H_n^{(1)}(kr)e^{i\phi} \right]. \]

Thus the lamellar and solenoidal components are given by

\[ v_r^E = -v_0 \left[ J_0^{(1)}(kr) + 2 \sum_{n=1}^{\infty} i^n [J_n^{(1)}(ka) + A_n] \frac{H_n^{(1)}(kr)}{H_n^{(1)}(ka)} \cos n\phi \right], \]

\[ v_\phi^E = \frac{2v_0}{kr} \sum_{n=1}^{\infty} i^n n J_n^{(1)}(ka) + A_n \frac{H_n^{(1)}(kr)}{H_n^{(1)}(ka)} \sin n\phi, \]

\[ v_r^\Omega = \frac{2v_0}{r} \sum_{n=1}^{\infty} i^n B_n \frac{H_n^{(1)}(kr)}{n H_n^{(1)}(ka)} \cos n\phi, \]

\[ v_\phi^\Omega = -2v_0 \kappa^2 \sum_{n=1}^{\infty} i^n \frac{B_n H_n^{(1)}(kr)}{n H_n^{(1)}(ka)} \sin n\phi. \]

Evidently, \( A_n = B_n \) \((n \geq 1)\) ensures that the additional normal velocities cancel. Then

\[ v_\phi^E(a, \phi) + v_\phi^\Omega(a, \phi) + v_\phi(a, \phi) = 0 \quad (\pi < \phi \leq \pi) \]

implies the coefficients are given by

\[ A_n = \frac{2}{\pi \kappa a H_n^{(1)}(ka)} \left( 1 - \frac{ka H_n^{(1)}(ka) \kappa a H_n^{(1)}(ka)}{n H_n^{(1)}(ka) n H_n^{(1)}(ka)} \right)^{-1} \]

\((n \gg 1)\).

III. THE ELLIPTIC CYLINDER

Elliptic coordinates \((\xi, \eta)\) are related to the Cartesian coordinates \((x, y)\) through \(x + iy = c \cosh(\xi + i\eta)\). Separation of variables in the Helmholtz equation (6) allows the solution to be written in terms of the Mathieu functions, with the parameter

\[ q = \frac{1}{4}(kc)^2, \]

in the notation of Jones, except for \(\exp(-i\omega t)\) time behavior instead of his \(\exp(i\omega t)\). The expansion of Eq. (11) corresponding to Eq. (13) is

\[ \exp[-ik(x \cos \phi + y \sin \phi)] = \exp[-i2q^{1/2}(\cosh \xi \cos \eta \cos \phi + \sinh \xi \sin \eta \sin \phi)] \]

\[ = 2 \sum_{m=0}^{\infty} (-i)^m c_m(\xi, \eta)c_m(\eta, \xi) M_m^{(1)}(\xi, \eta, q) \]

\[ + 2 \sum_{m=1}^{\infty} (-i)^m s_m(\xi, \eta)s_m(\eta, \xi) M_m^{(1)}(\xi, \eta, q), \]

in which it is noted that the functions of \(\eta\) also depend on \(q\). The elliptical scatterer boundary of Fig. 1 is the surface \( \xi = \xi_0 \) where

\[ \tanh \xi_0 = b/a, \quad c^2 = a^2 - b^2. \]

The deduction of a vector field
\[
\mathbf{F}(\xi, \eta) = \hat{\xi} F_\xi(\xi, \eta) + \hat{\eta} F_\eta(\xi, \eta)
\]
defined by
\[
\nabla \cdot \mathbf{F}(\xi, \eta) = \Xi(\xi)H(\eta), \tag{22}
\]
with
\[
\frac{d^2 \Xi(\xi)}{d \xi^2} + \left[ -\alpha + \frac{1}{2}(kc)\cosh 2\xi \right] \Xi(\xi) = 0,
\]
\[
\frac{d^2 H(\eta)}{d \eta^2} + \left[ \alpha - \frac{1}{2}(kc)^2 \cos 2\eta \right] H(\eta) = 0 \tag{23}
\]
is achieved by noting that
\[
\frac{\partial}{\partial \xi} [\Xi'(\xi)H(\eta)] + \frac{\partial}{\partial \eta} [\Xi(\xi)H'(\eta)]
= \Xi''(\xi)H(\eta) + \Xi(\xi)H''(\eta) = -k^2 h^2 \Xi(\xi)H(\eta) \tag{24}
\]
where the metric \( h(\xi, \eta) \) is given by
\[
 h = c(\cosh^2 \xi - \cos^2 \eta)^{1/2} = c \left[ \frac{1}{2}(\cosh 2\xi - \cos 2\eta) \right]^{1/2}.
\]
Comparison of Eq. (24) with the component form of Eq. (22), namely
\[
\frac{\partial}{\partial \xi} (hF_\xi) + \frac{\partial}{\partial \eta} (hF_\eta) = h^2 \Xi H, \tag{25}
\]
then indicates
\[
\mathbf{F}(\xi, \eta) = -\frac{1}{k^2 h(\xi, \eta)} \left[ \hat{\xi} \Xi'(\xi)H(\eta) + \hat{\eta} \Xi(\xi)H'(\eta) \right] \tag{26}
\]
which is irrotational. Similarly,
\[
(\nabla \times \mathbf{F}(\xi, \eta))_z = \Xi(\xi)H(\eta),
\]
that is
\[
\frac{\partial}{\partial \xi} (hF_\eta) - \frac{\partial}{\partial \eta} (hF_\xi) = h^2 \Xi H, \tag{27}
\]
has the solenoidal solution
\[
\mathbf{F}(\xi, \eta) = \frac{1}{k^2 h(\xi, \eta)} \left[ \hat{\xi} \Xi'(\xi)H'(\eta) - \hat{\eta} \Xi(\xi)H(\eta) \right]. \tag{28}
\]
The incident plane wave (11) has, by substitution of Eq. (21), the components
\[
u^{\xi}_0(\xi, \eta) = \frac{2v_0}{kh(\xi, \eta)} \left\{ \sum_{m=0}^{\infty} (-i)^{m} c_e(\phi_j, q) c_e(\eta, q)MC^{(1)}_m(\xi, q) \right. \\
+ \left. \sum_{m=1}^{\infty} (-i)^{m} s_e(\phi_j, q) s_e(\eta, q)MS^{(1)}_m(\xi, q) \right\}, \tag{29}
\]
\[
u^{\eta}_0(\xi, \eta) = \frac{2v_0}{kh(\xi, \eta)} \left\{ \sum_{m=0}^{\infty} (-i)^{m} c_e(\phi_j, q) c_e(\eta, q)MC^{(1)}_m(\xi, q) \right. \\
+ \left. \sum_{m=1}^{\infty} (-i)^{m} s_e(\phi_j, q) s_e(\eta, q)MS^{(1)}_m(\xi, q) \right\}.
\]
Equations (6) and (7) show that \( E \) and \( \Omega \) depend on the parameters
\[
q = \frac{1}{4}(kc)^2 = \frac{1}{4}(ka)^2[1 - (b/a)^2],
\]
\[
s = \frac{i\omega}{4v} c^2 = \frac{i}{4\varepsilon}(k_0a)^2[1 - (b/a)^2],
\]
respectively. Their solutions in terms of separated functions may be written as
\[
E(\xi, \eta) = 2v_0 k \sum_{m=0}^{\infty} \left[ A_m + M^{(1)}_m(\xi, \eta) c_e(\phi_j, q) \right] \\
\times (-i)^m c_e(\eta, q) \frac{MC^{(3)}_m(\xi, q)}{MC^{(3)}_m(\xi_0, q)} \\
+ 2v_0 k \sum_{m=1}^{\infty} \left[ B_m + M^{(1)}_m(\xi, \eta) s_e(\phi_j, q) \right] \\
\times (-i)^m s_e(\eta, q) \frac{MS^{(3)}_m(\xi, q)}{MS^{(3)}_m(\xi_0, q)}, \tag{32}
\]
\[
\Omega(\xi, \eta) = 2v_0 \kappa \sum_{m=0}^{\infty} \left[ C_m c_e(\eta, s) \frac{MC^{(3)}_m(\xi, s)}{MC^{(3)}_m(\xi_0, s)} \right] \\
+ 2v_0 \kappa \sum_{m=1}^{\infty} \left[ D_m s_e(\eta, s) \frac{MS^{(3)}_m(\xi, s)}{MS^{(3)}_m(\xi_0, s)} \right]. \tag{33}
\]
Use of Eq. (26) shows that the irrotational (lamellar) part of the scattered velocity has components
\[
u^{\xi}_0(\xi, \eta) = \frac{2v_0}{kh(\xi, \eta)} \left\{ \sum_{m=0}^{\infty} (-i)^{m} c_e(\phi_j, q) c_e(\eta, q)MC^{(1)}_m(\xi, q) \right. \\
+ \left. \sum_{m=1}^{\infty} (-i)^{m} s_e(\phi_j, q) s_e(\eta, q)MS^{(1)}_m(\xi, q) \right\}, \tag{29}
\]
\[
\times (-i)^m s e_m(\eta,q) \frac{M c_m^{(3)}(\xi,q)}{M c_m^{(3)}(\xi,0)} \bigg\}, \quad (34)
\]

in which the terms with inviscid structure are identified. Similarly, use of Eq. (28) shows that the solenoidal part of the scattered velocity has components

\[
v^{(3)}_v(\xi,\eta) = \frac{2v_0}{k(h(\xi,\eta))} \left\{ \sum_{m=0}^{\infty} C_m c_m(\eta,s) \frac{M c_m^{(3)}(\xi,s)}{M c_m^{(3)}(\xi,0)} \right\} + \sum_{m=1}^{\infty} D_m s e_m(\eta,s) \frac{M c_m^{(3)}(\xi,s)}{M c_m^{(3)}(\xi,0)} \bigg\}, \quad (35)
\]

The no-slip boundary is now enforced by setting

\[
v^{(3)}_v(\xi_0,\eta) + v^{(3)}_v(\xi_0,\eta) + v^{(3)}_v(\xi_0,\eta) = 0, \quad (36)
\]

over the entire elliptical surface where \(-\pi \leq \eta \leq \pi\). The even (in \(\eta\)) part of the \(\xi\) component of this boundary condition is

\[
\sum_{m=0}^{\infty} \left[ A_m + M c_m^{(1)}(\xi_0,q) c e_m(\phi_1,q) \right] (-i)^m c e_m(\eta,q) \]

\[
= \sum_{m=0}^{\infty} (-i)^m c e_m(\phi_1,q) c e_m(\eta,q) M c_m^{(1)}(\xi_0,q). \quad (37)
\]

The odd (in \(\eta\)) part of the \(\xi\) component of Eq. (36) is

\[
\sum_{m=1}^{\infty} \left[ B_m + M s_m^{(1)}(\xi_0,q) s e_m(\phi_1,q) \right] (-i)^m s e_m(\eta,q) \]

\[
= \sum_{m=1}^{\infty} (-i)^m s e_m(\phi_1,q) s e_m(\eta,q) M s_m^{(1)}(\xi_0,q). \quad (38)
\]

The odd (in \(\eta\)) part of the \(\eta\) component of Eq. (36) is

\[
\sum_{m=0}^{\infty} \left[ A_m + M c_m^{(1)}(\xi_0,q) c e_m(\phi_1,q) \right] \times (-i)^m c e_m'(\eta,q) \frac{M c_m^{(3)}(\xi_0,q)}{M c_m^{(3)}(\xi_0,q)} \]

\[
= \sum_{m=0}^{\infty} (-i)^m c e_m(\phi_1,q) c e_m'(\eta,q) M c_m^{(1)}(\xi_0,q). \quad (39)
\]

The even (in \(\eta\)) part of the \(\eta\) component of Eq. (36) is

\[
\sum_{m=1}^{\infty} \left[ B_m + M s_m^{(1)}(\xi_0,q) s e_m(\phi_1,q) \right] \times (-i)^m s e_m'(\eta,q) \frac{M s_m^{(3)}(\xi_0,q)}{M s_m^{(3)}(\xi_0,q)} \]

\[
= \sum_{m=1}^{\infty} (-i)^m s e_m(\phi_1,q) s e_m'(\eta,q) M s_m^{(1)}(\xi_0,q). \quad (40)
\]

At this stage, major difficulties associated with the elliptic coordinates are encountered. The periodic functions have the expansions

\[
ce_{2m+1}(\eta,q) = \sum_{j=0}^{\infty} \alpha_{2m+1,q} \cos(2j+\eta) \]

\((m = 0, 1, 2, \ldots, p = 0, 1)\),

\[
s e_{2m+1}(\eta,q) = \sum_{j=0}^{\infty} \beta_{2m+1,q} \sin(2j+\eta) \quad (41)
\]

\((m = 0, 1, 2, \ldots, p = 0, 1)\),

and have the orthogonality properties

\[
\int_{-\pi}^{\pi} c e_m(\eta,q) c e_n(\eta,q) d\eta = \pi \delta_{mn} \quad (m,n \geq 0),
\]

\[
\int_{-\pi}^{\pi} s e_m(\eta,q) s e_n(\eta,q) d\eta = \pi \delta_{mn} \quad (m,n \geq 1), \quad (42)
\]

but, in contrast to \(\cos n \eta\) and \(\sin n \eta\), the derivative of one is not a multiple of the other, the derivatives are not orthogonal and they depend on the parameter \(q\), giving different sets of periodic functions in \(E\) and \(\Omega\). Inspection of Eqs. (37)–(40) shows that the introduction of the inner products

\[
\Lambda_{m,n}(q,s) = \langle c e_m(q), s e_n'(s) \rangle
\]

\[
= \int_{-\pi}^{\pi} c e_m(\eta,q) s e_n'(\eta,s) d\eta \quad (m > 0, n > 1) \quad (43)
\]

suffices to exploit the orthogonality (42) because integration by parts and the periodicity of the angular Mathieu functions yields
\[-\Lambda_{m,n}(q,s) = \left\langle ce_m'(q), se_n(q) \right\rangle = \int_{-\pi}^{\pi} ce_m'(\eta,q)se_n(\eta,s)d\eta \quad (m \geq 0, n \geq 0).\]

(44)

This inner product is zero if \(m\) and \(n\) are of different parity and, although the functions are complex-valued, does not involve any complex conjugation. In terms of the coefficients in Eq. (41),

\[
\int_{-\pi}^{\pi} ce_{2n+p}(\eta,q)se_{n}(\eta,s)d\eta = \pi \delta_{pr} \sum_{j=0}^{\infty} A_{2n+p}(q)B_{j}(2j+p) \quad (m,n \geq 0, p,r = 0 \text{ or } 1,n+r \neq 0).
\]

(45)

Multiplication of Eqs. (37)–(40) by \(ce_n(\eta,q), se_n(\eta,q), se_n(\eta,s), ce_n(\eta,s)\), respectively, and subsequent integration of each from \(\eta=-\pi\) to \(\eta=\pi\) yields the sets of linear equations

\[
A_n = \frac{1}{\pi (-i)^{n}} k \sum_{m=1}^{\infty} D_{m}\Lambda_{mn}(q,s) \quad (n \geq 0),
\]

(46)

\[
B_n = -\frac{1}{\pi (-i)^{n}} \sum_{m=0}^{\infty} C_{m}\Lambda_{mn}(s,q) \quad (n \geq 1),
\]

(47)

\[
\frac{k}{\kappa} D_n = \frac{M_{n}^{(3)}(\xi,0,s)}{\pi M_{s}^{(3)}(\xi,0,s)} \sum_{m=0}^{\infty} \Lambda_{mn}(q,s)(-i)^{m} \times \left[ A_{n}^{m} \frac{M_{m}^{(3)}(\xi,0,q)}{M_{m}^{(3)}(\xi,0,q)} + \frac{2ce_m(\phi,0,q)}{i\pi M_{m}^{(3)}(\xi,0,q)} \right] \times (n \geq 1),
\]

(48)

\[
\frac{k}{\kappa} C_n = \frac{2se_{m}(\phi,0,q)}{i\pi M_{m}^{(3)}(\xi,0,q)} \sum_{m=1}^{\infty} \Lambda_{nm}(s,q)(-i)^{m} \times \left[ B_{n}^{m} \frac{M_{m}^{(3)}(\xi,0,q)}{M_{m}^{(3)}(\xi,0,q)} + \frac{2se_m(\phi,0,q)}{i\pi M_{m}^{(3)}(\xi,0,q)} \right] \quad (n \geq 0),
\]

(49)

after use of the Wronskians

\[
M_{m}^{(1)}(\xi,0,q)M_{m}^{(3)}(\xi,0,q) - M_{m}^{(1)}(\xi,0,q)M_{m}^{(3)}(\xi,0,q) = \frac{2}{\pi i},
\]

(50)

and similarly,

\[
M_{m}^{(1)}(\xi,0,q)M_{m}^{(3)}(\xi,0,q) - M_{m}^{(1)}(\xi,0,q)M_{m}^{(3)}(\xi,0,q) = \frac{2}{\pi i}.
\]

The two disjoint pairs of linear systems may be written symbolically as

\[
\begin{bmatrix}
[I] & [Q] & [A] \\
-[R] & [I] & [D]
\end{bmatrix} =
\begin{bmatrix}
[0] \\
[F]
\end{bmatrix},
\]

(51)

or, if elimination of one set of coefficients from each is preferred,

\[
\begin{bmatrix}
[I] - [R][P][D] = [F],
\end{bmatrix} \quad (52)
\]

\[
\begin{bmatrix}
[I] - [S][Q][C] = [G].
\end{bmatrix} \quad (53)
\]

The infinite series are truncated at suitably large indices, such that the results exhibit satisfactory numerical or “self” convergence.

The definition (31) indicates the need for an approximation to the modified Mathieu function of large imaginary parameter \(q=\imath|q|\) with \(|q| \to \infty\). The first equation of Eq. (23) has the form

\[
\frac{df(\xi)}{d\xi^2} + (2q \cosh 2\xi - \alpha)f(\xi) = 0,
\]

(54)

whose solutions are of four types, even/odd functions of even/odd orders. McLachlan\(^{19}\) gives asymptotic forms for \(\alpha\) as \(|q| \to \infty\). For example, with \(q=\imath|q|\),

\[
\alpha_{2n} \sim -2|q| + (8n + 2)|q|^{1/4}e^{i\pi/4},
\]

(55)

and application of the WKB(J) or LG method to Eq. (54) eventually yields

\[
f(\xi) \sim \frac{1}{\sqrt{\cosh \xi}} \exp\left\{-\left(1 - i\right)|q|^{1/2}\sinh \xi \right\} + (2n + 1/2)i \tan^{-1}(\sinh \xi),
\]

(57)

which is a decaying solution, with \(f(0)=1\). Evidently \(f'(\xi)\) has a dominant term of \(O(|q|^{1/2})\).

According to p. 240 of McLachlan,\(^{19}\) the even order modified Mathieu functions of the cosine type and the odd order functions of the sine type have the same asymptotic eigenvalues, and thus, as \(|q| \to \infty\),

\[
M_{2n}^{(3)}(\xi,\imath|q|) \sim M_{2n+1}^{(3)}(\xi,\imath|q|) \sim \frac{1}{\sqrt{\cosh \xi}} \exp\left\{-\left(1 - i\right)|q|^{1/2}\sinh \xi \right\} + (2n + 1/2)i \tan^{-1}(\sinh \xi),
\]

(58)

and similarly,

\[
M_{2n+1}^{(3)}(\xi,\imath|q|) \sim M_{2n}^{(3)}(\xi,\imath|q|) \sim \frac{1}{\sqrt{\cosh \xi}} \exp\left\{-\left(1 - i\right)|q|^{1/2}\sinh \xi \right\} + (2n + 3/2)i \tan^{-1}(\sinh \xi).\]

(59)

The system of linear equations (51) requires the ratios

\[
\frac{M_{n}^{(3)}(\xi,\imath|q|) - M_{n}^{(3)}(\xi,\imath|q|)}{M_{n}^{(3)}(\xi,\imath|q|) - M_{n}^{(3)}(\xi,\imath|q|)} \sim \frac{\text{sech} \xi}{(i-1)|q|^{1/2}},
\]

(60)

without regard for either the type or the parity of the order. By calling the eigenvalue solver once only for a given \(q\), the
Mathieu function calculations are fast.\textsuperscript{16}

IV. FAR SCATTERED FIELD

For the circle, substitution of the asymptotic form

\[ H_n^{(1)}(kr) \sim \sqrt{\frac{2}{\pi kr}} k^{-n} e^{ikr} \]

in Eq. (16) demonstrates that the vorticity \( \Omega \) is limited to a thin boundary layer on the surface of the scatterer. Since \( v \) and \( p \) are related by Eq. (10), the single scalar function of interest in the far field is the divergence \( E \), which is simply proportional to the acoustic pressure field and given by Eq. (15). In the far field,

\[ E(r \to \infty, \phi) = v_0 k \sqrt{\frac{2}{\pi kr}} e^{ikr} F(\phi), \]

(61)

where

\[ F(\phi) = \sum_{n=-\infty}^{\infty} \left[ i^n J_n^\prime(ka) + \frac{A_n}{H_n^{(1)}(ka)} \right] e^{in\phi}. \]

(62)

This separation of the radial and angular factors facilitates the graphical display of the scattering pattern \( F(\phi) \).

For the ellipse, \( c e^{i\phi}/2 \to r \to \infty, \eta \to \phi \) in the far field and the required asymptotic forms, deduced from the first equation of Eq. (23), are\textsuperscript{18}

\[ M_n^{(1)}(\xi, q) \sim \sqrt{\frac{2}{\pi k r^3}} e^{ikr\eta}(\xi, q), \]

(63)

The asymptotic forms (58) and (59) verify the exponential decay of the vorticity. From Eq. (32), the scattering pattern is given by

\[ F(\phi) = 2 \sum_{m=0}^{\infty} A_m + M_1^{(1)}(\xi_0, q) c e_n(\phi_i, q) \] \[ + 2 \sum_{m=1}^{\infty} B_m + M_1^{(1)}(\xi_0, q) s e_n(\phi_i, q) \]

\[ \times (-i)^m \frac{c e_n(\phi_i, q)}{M_1^{(1)}(\xi_0, q)}. \]

(64)

V. THE STRIP

The incident plane wave defined by Eq. (11) requires the scattered field on the surface of the strip \( (b \to 0) \) in Fig. 1) to satisfy

\[ v(\chi, 0) = i v_0 (\hat{x} \cos \phi + \hat{y} \sin \phi) e^{-ikx \cos \phi} \quad (|x| < a). \]

(65)

The solution structure follows the Wiener-Hopf analysis of Davis and Nagem\textsuperscript{2} as far as the imposition of the boundary conditions at \( y=0 \). As in these authors' subsequent aperture\textsuperscript{5} and disk\textsuperscript{3} studies in 3D, a method suited to creeping flow is then adopted and therefore passage to the inviscid limit is precluded. Functional forms displaying the square root edge singularity are posed for the pressure and tangential stress discontinuities across the strip and their coefficients determined by application of Eq. (65).

On defining dimensionless Fourier transforms

\[ [V(\alpha, y), P(\alpha, y)] = \int_{-\infty}^{\infty} k_0 \left[ \frac{v(x, y)}{v_0}, \frac{p(x, y)}{\rho_0 c_0 v_0} \right] e^{-i\alpha x} dx, \]

(66)

the continuity equation gives

\[ i \alpha V_{\xi}(\alpha, y) + \frac{\partial V_{\eta}(\alpha, y)}{\partial y} = \frac{i \omega}{c_0} P(\alpha, y) \]

(67)

and the momentum equation yields

\[ \left( \frac{d^2}{dy^2} + \frac{i \omega}{\nu} - \alpha^2 \right) V_{\eta}(\alpha, y) = \frac{c_0}{\nu} \left( 1 - \frac{i \epsilon^2}{3} \right) \left[ \frac{i \alpha P(\alpha, y)}{\frac{\partial}{\partial y} P(\alpha, y)} \right]. \]

(68)

The Helmholtz equation (6) implies

\[ \left( \frac{d^2}{dy^2} + k^2 - \alpha^2 \right) P(\alpha, y) = 0, \]

(69)

whose solution is

\[ P(\alpha, y) = [A(\alpha) + B(\alpha) \text{sgn } y] \exp \{- (\alpha^2 - k^2)^{1/2} |y|\}. \]

(70)

Hence the scaled Fourier transform of the velocity field is

\[ V(\alpha, y) = \begin{bmatrix} \frac{C(\alpha) + D(\alpha) \text{sgn } y}{(\omega^2 - i \omega \nu)^{1/2}} \\times \exp \{- (\alpha^2 - k^2)^{1/2} |y|\} \]

\[ \frac{k_0}{\nu} \left[ - \frac{i\alpha[A(\alpha) + B(\alpha) \text{sgn } y]}{k^2} \right] \exp \{- (\alpha^2 - k^2)^{1/2} |y|\}. \]

(71)

The continuity of \( V \) gives

\[ D(\alpha) + \frac{k_0 \alpha}{k^2} B(\alpha) = 0; \]

(72)

\[ \frac{\alpha}{(\alpha^2 - i \omega \nu)^{1/2}} C(\alpha) + \frac{k_0}{k} (\alpha^2 - k^2)^{1/2} A(\alpha) = 0, \]

(73)

whence

\[ V(\alpha, 0) = \begin{bmatrix} \frac{1}{\alpha^2 (\alpha^2 - k^2)^{1/2}} C(\alpha) \\frac{i}{(\alpha^2 - k^2)^{1/2}} D(\alpha) \\frac{\alpha}{(\alpha^2 - i \omega \nu)^{1/2}} - i \frac{D(\alpha)}{\alpha} \end{bmatrix}, \]

(74)

where
\[ K(\alpha) = (\alpha^2 - k^2)^{1/2} - \frac{\alpha^2}{(\alpha^2 - i\omega \nu)^{1/2}} \]  

(75)

is the Wiener-Hopf kernel of Davis and Nagem.\(^2\) Also, the jump in the normal derivative of \( V \) is given by

\[
\frac{1}{2} \left[ \frac{dV(\alpha,y)}{dy} \right]_{y=0^+} = \left[ \frac{i\omega \nu}{(\alpha^2 - i\omega \nu)^{1/2}} C(\alpha) + \frac{ik^2}{\alpha} D(\alpha) \right].
\]  

(76)

after substitution of Eqs. (72) and (73).

The discontinuity in normal stress across the strip has square root edge singularities and is conveniently represented as

\[
\text{[normal stress]}_{y=0^+} = \rho_0 c_0 v_0 \left[ \frac{\partial p(x,y)}{\partial y} \right]_{y=0^+} \left[ 1 + \frac{2i}{3} e^2 \right] + \frac{2\nu k_0}{\omega v_0} \frac{\partial v_y(x,y)}{\partial y} \right|_{y=0^+} = 2\rho_0 c_0 v_0 \sum_{n=0}^{\infty} \epsilon_n n^2 a_n T_n(x/a) \sqrt{1 - (x/a)^2} \]  

(77)

Then the Fourier coefficient

\[
\int_{-a}^{a} T_n(x/a) e^{-i\alpha x} dx = \pi a (-i)^n J_n(aa)
\]  

(78)

enables the Fourier transform of Eq. (77) to be written as

\[
\pi a \sum_{n=0}^{\infty} \epsilon_n a_n J_n(aa) = \frac{D(\alpha)}{\alpha},
\]  

(79)

after substitution of Eqs. (8), (9), and (72). Similarly, the representation

\[
\text{[tangential stress]}_{y=0^+} = \rho_0 c_0 v_0 \left[ \frac{\partial v_x(x,y)}{\partial y} + \frac{\partial}{\partial x} v_y(x,y) \right]_{y=0^+} = 2\rho_0 c_0 v_0 \sum_{n=0}^{\infty} \epsilon_n n b_n T_n(x/a) \sqrt{1 - (x/a)^2} \]  

(80)

has the Fourier transform

\[
\pi a \sum_{n=0}^{\infty} \epsilon_n b_n J_n(aa) = \left[ \frac{\nu}{2\alpha} \frac{d}{dy} V_x(\alpha, y) \right]_{y=0^+} = \frac{i}{(\alpha^2 - i\omega \nu)^{1/2}} C(\alpha),
\]  

(81)

after substitution of Eqs. (76) and (78).

Equations for the coefficients \( \{a_n, b_n; n \geq 0\} \) are now established by enforcing the no-slip condition (65). For \( |x| < a \), Eqs. (66) and (74) yield

\[
iv \left[ \cos \phi_1 \frac{e^{-ikx}}{\sin \phi_1} \right] = \frac{v(x,0)}{v_0} e^{i\alpha x} = \frac{1}{2\pi k_0} \int_{-\infty}^{\infty} [V(\alpha,0)] e^{i\alpha x} d\alpha
\]  

(82)

whose Chebyshev coefficients are determined by applying the operator

\[
\frac{i^m}{\pi a} \int_{-a}^{a} T_m(x/a) \sqrt{1 - (x/a)^2} dx \quad (m \geq 0).
\]

Thus, by using Eq. (78) and substituting Eqs. (79) and (81),

\[
\left[ \cos \phi_1 \right] J_n(ka \cos \phi_1) = \left[ \frac{D(\alpha)}{\alpha} \right] \int_{-\infty}^{\infty} K(\alpha) \sum_{n=0}^{\infty} \epsilon_n a_n J_n(aa) \sqrt{1 - (x/a)^2} \]  

(83)

since \( K(\alpha) \) and the factor \( (\cdot)^{1/2} \) are even functions of \( \alpha \).

From Eq. (80), the vorticity discontinuity across the plate is

\[
\left[ \Omega(x,y) \right]_{y=0^+} = -\frac{\partial}{\partial y} v_x(x,y) = -\frac{k_0 v_0}{\epsilon} \sum_{n=0}^{\infty} \epsilon_n b_n T_n(x/a) \sqrt{1 - (x/a)^2}
\]  

(84)

while the continuous vorticity at the strip is the continuous part of

\[
\frac{v_0}{2\pi k_0} \int_{-\infty}^{\infty} \left( i\alpha V_x(\alpha,0) - \frac{dV_x}{dy}(\alpha,0) \right) e^{i\alpha x} d\alpha
\]  

at the strip \( (|x| < a) \), which is
FIG. 2. Plane wave excitation of elliptic cylinder. Case: $k_0 a = 5$, $b/a = 0.1$, $\epsilon = 0.1$, and $\phi_i = 0$. (a) Surface vorticity $\Omega(\xi_i, \eta)$ and (b) scattering pattern.

\[
\begin{align*}
\frac{v_0}{2 \pi k_0} & \int_{-\infty}^{\infty} \left[ K(\alpha) D(\alpha) + (\alpha^2 - i \omega / v)^{1/2} B(\alpha) \right] e^{i \alpha x} d\alpha = \frac{v_0}{2 \pi k_0} (-i \omega / v) \pi a \\
& \times \int_{-\infty}^{\infty} \frac{\epsilon_n^{i \alpha \alpha}}{\alpha^2 - i \omega / v} \sum_{n=0}^{\infty} \epsilon_n a_n f_n(\alpha a) d\alpha \\
& = \frac{\omega a v_0}{k_0 v} \int_{0}^{\infty} \frac{\alpha d\alpha}{(\alpha^2 - i \omega / v)^{1/2}} \sum_{n=0}^{\infty} \epsilon_n a_n f_n(\alpha a) \\
& \times \left\{ \begin{array}{ll}
\sin \alpha \alpha & (n \text{ even}) \\
-i \cos \alpha \alpha & (n \text{ odd})
\end{array} \right. \\
\end{align*}
\]

(85)

after substitution of Eqs. (71), (72), (75), and (79). For comparison with the ellipse and circle, the vorticity on either side of the strip is deduced from Eqs. (84) and (85). The normalization $\Omega / v_0 k$ is used in the graphs.

VI. RESULTS

The acoustical size of the cylinder characterized by $k_0 a = 5$ is neither small (quasistatic regime) nor large (ray acoustics regime) in terms of the wavelength of the exciting plane wave, and therefore this scatterer is truly in the “resonant” frequency range. The dimensionless viscosity parameter of Eq. (9) is fixed at $\epsilon = 0.1$. An extremely flattened elliptic cylinder having $b/a = 0.1$ (ellipticity $e = [1 - (b/a)^2]^{1/2}$ = 0.995) is selected, in Figs. 2 and 3, to demonstrate the variations in the scattering with plane-wave incidence angle. Figure 2 is for incidence from the +x axis, that is, from the azimuthal angle $\phi_i = 0$, which is the “endfire” direction. Figure 3 is for incidence from the +y axis, that is, from the azimuthal angle $\phi_i = \pi / 2$, which is the “broadside” direction. Figures 2(a) and 3(a) depict the complex-valued surface vorticity $\Omega(\xi_i, \eta)$, whose driving mechanism is the tangential component of the incident field. This is analogous to the unidirectional viscous flow known as a Stokes wave. As expected, endfire incidence generates vorticity along the narrow ellipse but broadside incidence yields vorticity confined near the ends. Viscosity effects on the far field are displayed in Figs. 2(b) and 3(b), which graph, for the two cases, the scattering pattern $|F(\phi)|$ of Eq. (64). Viscosity is clearly more influential far from the scatterer when the incident acoustic wave is essentially grazing to the elongated obstacle. Note that in the limiting case of the hard strip ($b/a \to 0$) in an inviscid medium, no scattered field is produced when $\phi_i = 0$. The elongated ellipse still presents some disturbance to the field generated by the endfire source, but the scattering pattern curve of Fig. 2(b) is an order-of-magnitude less than the curve of Fig. 3(b).

As $b/a \to 1$ the ellipse degenerates to the circle, and Figs. 4(a) and 4(b) present the companion curves for this case. The numerical values from the analysis of Sec. III in
The presence of a slight medium viscosity can have a substantial effect on the scattering pattern of an impenetrable elliptic cylinder, even though the viscosity-induced vorticity is confined to a narrow boundary layer on the ellipse. Unlike the inviscid case, the forward scatter in a viscous medium can be significant when a plane wave strikes an elongated ellipse from the mostly grazing direction. The analysis and its implementation are validated via the agreement between the Mathieu function calculations for the elliptic cylinder and the independent calculations for the limiting cases of the circle and the strip. Though presented as a fundamental study, the results provide an obvious warning that endfire sonar detection of slender bodies can be unreliable.

VII. CONCLUSIONS

The presence of a slight medium viscosity can have a substantial effect on the scattering pattern of an impenetrable elliptic cylinder, even though the viscosity-induced vorticity is confined to a narrow boundary layer on the ellipse. Unlike the inviscid case, the forward scatter in a viscous medium can be significant when a plane wave strikes an elongated ellipse from the mostly grazing direction. The analysis and its implementation are validated via the agreement between the Mathieu function calculations for the elliptic cylinder and the independent calculations for the limiting cases of the circle and the strip. Though presented as a fundamental study, the results provide an obvious warning that endfire sonar detection of slender bodies can be unreliable.