DECOMPOSITION OF THE SCATTERING BY A FINITE LINEAR ARRAY INTO PERIODIC AND EDGE COMPONENTS

Douglas R. Denison
Department of Electrical Engineering and Computer Science
Massachusetts Institute of Technology
77 Massachusetts Avenue
Cambridge, Massachusetts 02139

Robert W. Scharstein
Department of Electrical Engineering
The University of Alabama
317 Houser Hall
Tuscaloosa, Alabama 35487

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Phased arrays, grating, edge waves, edge effects, scattering

ABSTRACT
The field scattered by each element in a finite array is decomposed into the contribution of the infinite or periodic array, a component due to the presence of the left edge of the array, and a component due to the right edge of the array. The scattered field due to either array edge is derived from a separate analysis of the appropriate semi-infinite array and is interpreted in terms of a decaying wave that is launched from the array end. © 1995 John Wiley & Sons, Inc.

I. INTRODUCTION
The influences of edge effects and coupling on scattering by finite arrays are studied in this work via the analysis and interpretation of a linear grating of conducting circular cylinders. The radius of these identical thin cylinders is considered to be small compared to a wavelength, so the cylinder does not see the wave nature of the incident field and thus the uniform current distribution on each cylinder surface can be represented by a single complex number. This is the problem of a grating of wires, where each wire is infinitely long and parallel to all other wires in the array. Therefore, the problem of a normally incident plane wave is entirely two dimensional and there is no variation of any physical quantity in the axial direction. This is probably the simplest geometry imaginable wherein the pertinent physics in the plane of a finite array or grating is captured. Hansen and Gammon [1] effectively achieve similar simplification by considering a finite-by-infinite array of dipoles.

This article extends the work in [1] by providing a physically based ansatz to explicitly compute the edge-wave contributions to the total current on each element. Usoff and Munk [2] also analyze the case of a finite-by-infinite array of thin wire elements, and their general approach is to also examine edge contributions (via UTD for a related strip) on top of the bulk array contribution (via Floquet currents). In contrast to the moderately complex, element-dependent reaction integrals in [2], the present article fully exploits the single-mode geometry, whereby the desired edge-wave interactions are intuitively and accurately dissected and then reassembled. The functional dependence on the array grid (element spacing $kd$) and angle of incidence ($\phi_i$) is traceable at each stage of the mathematical development, which requires only linear algebra and series summation. Hence, the role of the (individually) isotropic elements is minimized, giving a clear picture of the interaction between the original excitation and this important canonical array. Incidentally, the choice of plane-wave incidence, that is, the scattering problem, is a matter of conceptual convenience, and is no different (except for a minus sign in the forcing term) than the phase-scanned radiation problem for this simple geometry. The contribution of the present article is the demonstration of the high accuracy and applicability of the edge-wave decomposition, even for very small arrays and for incidence (or scan) angles approaching grazing.

Infinite (or periodic) arrays are important geometries to antenna engineers [3], and the mathematical analysis of the semi-infinite array has received attention [4–6]. A study of the variation of the surface currents on electrically thick cylinders (multimode elements) as a function of position in various-sized finite arrays is summarized in [7]; for thin wires with $ka \sim 0.1$, the next azimuthal Fourier coefficient [of $\exp(\pm i\phi)$] is always at least three orders of magnitude less than the constant term for elements spaced by $\lambda/2$ or so. Using verifiably accurate computations, this article shows how the electromagnetic behavior of the finite, semi-infinite, and periodic arrays of simple, single-mode isotropic elements are physically related.

The set of element currents in the finite array is innately governed by a Toeplitz matrix equation. A particularly noteworthy feature of this simple geometry is that Floquet's theorem appears naturally in the mathematics, without the need for any external arguments, and allows a straightforward analysis of the periodic grating in terms of a reference current coefficient. The currents for the semi-infinite array are written as the sum of a periodic term and an edge-effect term that decays rapidly from the terminating edge. The subsequent semi-infinite series is then properly truncated, and the edge contributions are calculated. The analysis then proceeds to a superposition and decomposition of the finite array into its periodic and semi-infinite constituents. Resultant aperture distributions (sets of total wire currents) are graphed and physically interpreted.

II. THE FINITE ARRAY
Consider a finite array of $N$ infinitely long parallel cylinders with radius $a$ and uniform spacing $d$, illuminated by an incident plane wave $[\exp(-i\omega t)$ time convention$]$ of unit amplitude

$$\psi'(\rho, \phi) = e^{-k_0 \rho \cos(\phi - \phi_i)}, \quad (1)$$

where $\phi_i$ is the incidence angle of the wave as measured from the array, and $\rho$ is the radial distance measured from the first element of the array (Figure 1). The polarization of the electric field is $z$ directed, or TM, and the wave function $\psi$ represents the single component $E_z$. In this analysis, the
Dirichlet boundary condition holds on the surface of each wire:

$$\psi'(a, \phi) + \psi'(a, \phi) = 0 \quad (0 \leq \phi \leq 2\pi). \tag{2}$$

A weighted modal series of Hankel functions of the first kind represents the outward-traveling cylindrical waves of the scattered field:

$$\psi'(r, \phi) = \sum_{n=0}^{N-1} a_n H_0^{(1)}(kd|\mathbf{p} - \mathbf{p}_n|), \tag{3}$$

where the index $n$ denotes the $n$th member of the array (with the zeroth element marking the left end of the array) and $p_n$ is the distance to the $n$th wire from the origin. The $a_n$ is a weighting factor that determines the contribution of the $n$th element to the scattered field, and is a constant multiple of the axial surface current (discontinuity in normal derivative of $\psi$)

$$a_n = \frac{\imath \eta}{H_1^{(1)}(ka)} f(n), \tag{4}$$

where $\eta$ is the free-space impedance.

Before imposing the condition (2) on the incident and scattered fields, an observation about the interaction between elements is in order. If the element separation $d$ is at least an order of magnitude greater than the wire radius $a$, then, to first order, each wire can consider the other wires in the array as infinitesimally thin. Therefore, under this assumption, the mutual interaction between elements is a function of separation distance only and not of the element size. In other words, the azimuthally constant current on each wire surface can be represented by an equivalent filamentary current along the axis of each wire. Similarly, when computing the self-contribution to the surface field at the radius $a$, the common thin-wire kernel approximation applies. Computations made without this assumption [7] and based on a full azimuthal Fourier series give resultant currents that are practically indistinguishable from the present results, in the length-scale ranges $ka \leq 0.1$ and $kd \geq 1$.

Thus, enforcing the Dirichlet boundary condition (2) at the surface of the $m$th element gives the Toeplitz matrix equation

$$a_m H_0^{(1)}(ka) + \sum_{n=0}^{N-1} a_n H_0^{(1)}(kd|m - n|) = -e^{-\imath m kd \cos \phi}, \quad m = 0, 1, 2, \ldots, N - 1. \tag{5}$$

The prime (') denotes that $n \neq m$. For $ka$ small, the magnitude of $H_0^{(1)}(ka)$ is large, and the coefficient matrix will have diagonal dominance and thus will invert nicely. In all subsequent analysis, the value of $ka$ is taken to be 0.1. As a check for accuracy, the resultant $a_n$ were used in (3) to calculate the total field on the surface of each wire which, according to the boundary condition (2), should be zero. For the single precision arithmetic used with an incident-field magnitude of unity, and for several values of $N$, $kd$, and $\phi_0$, the surface electric field is consistently on the order of $10^{-9}$.

III. THE PERIODIC ARRAY

The Dirichlet boundary condition on each single-mode element in the periodic (or infinite) array is

$$\sum_{m=-\infty}^{\infty} b_n Z_{mn} = -e^{-\imath m kd \cos \phi}, \tag{6}$$

where

$$Z_{mn} = \begin{cases} H_0^{(1)}(kd|m - n|), & m \neq n, \\ H_0^{(1)}(ka), & m = n. \end{cases} \tag{7}$$

Multiplying both sides of (6) by $\exp(\imath m kd \cos \phi)$ and introducing the pair of canceling exponentials $\exp(\pm \imath kd \cos \phi)$ in the summation yields

$$\sum_{n=-\infty}^{\infty} b_n e^{\imath m kd \cos \phi} Z_{mn} e^{\imath (m - n) kd \cos \phi} = -1 \quad g(m - n) = 0, \pm 1, \pm 2, \ldots, \tag{8}$$

which is a discrete convolution. The only way this convolution between a nonconstant sequence $g$ and some arbitrary sequence $f$ can be constant for all $m$ is if the sequence $f$ is itself a constant. This fact implies that each coefficient $b_n$ is related to the $n = 0$ coefficient by

$$b_n = b_0 e^{-\imath n kd \cos \phi}, \tag{9}$$

which is a direct observation of the familiar Floquet phase shift of array theory.

Using (7) and (9) in evaluating (6) at the reference $(m = 0)$ element gives

$$b_0 \left[ H_0^{(1)}(ka) + 2 \sum_{n=1}^{\infty} \cos(nkd \cos \phi_0) H_0^{(1)}(nkd) \right] = -1. \tag{10}$$

Solving for the $b_0$ entails evaluating the slowly convergent series, which can be accelerated via the Kummer transform, whereby the asymptotic form of the summand is subtracted from and added to the original series. It follows from the principal large-argument form of the Hankel function that the sum needed for the Kummer transform of (10) is of the form

$$\sqrt{\frac{2}{\imath \pi kd}} \sum_{n=1}^{\infty} \frac{e^{\imath nkd(1 \pm \cos \phi_0)}}{\sqrt{n}}. \tag{11}$$

Evaluation of this series is given in Appendix A. Note that this series is divergent when $kd(1 \pm \cos \phi_0)$ is an integral multiple of $2\pi$—the condition for Woods' anomalies [8].

MICROWAVE AND OPTICAL TECHNOLOGY LETTERS / Vol. 9, No. 6, August 20 1995 339
In a large finite array, the center elements are virtually unaware of the existence of the left and right edges of the array, and thus the coefficients (currents) at the center of a large array should approximate those in an infinite array. Using this notion the periodic array coefficient can be checked against the coefficients from the center of a large finite array, which, by satisfying the Dirichlet boundary condition, have already been shown to be accurate. Tables 1 and 2 compare the coefficients of the central elements of an array of 50 wires to the periodic coefficient, and show good agreement in both magnitude and phase.

IV. THE SEMINIFINITE ARRAY

The geometry of the seminfinite array is similar to the array in Figure 1, with the number of elements ($N$) on the right end going to infinity. Because the array is essentially periodic far from the edge, the coefficients on wires far from the edge should approach a value equal to that of the wire coefficients in the periodic grating. Because of this far-from-the-edge behavior, the coefficients on the elements in the seminfinite array can be written as the sum of an infinite-array term, with appropriate phase factor, and an edge effect term [4, 5]:

$$ c_n = C_n + b_n e^{-i k d \cos \phi_i}, \quad (12) $$

Obviously, the perturbation coefficients $C_n$ should decay as the distance from the edge increases, thus rendering their contribution to the element currents negligible for large $n$.

The Dirichlet boundary condition on the elements of the seminfinite array is

$$ \sum_{m=0}^{\infty} c_n Z_{mn} = -e^{-i k d \cos \phi_i}, \quad m = 0, 1, 2, \ldots \ldots \quad (13) $$

Using (12) for the $c_n$ and subtracting out the periodic term yields

$$ \sum_{m=0}^{\infty} C_n Z_{mn} = -e^{-i k d \cos \phi_i} - b_n \sum_{m=0}^{\infty} e^{-i k d \cos \phi_i} Z_{mn}, $$

$$ m = 0, 1, 2, \ldots \ldots \quad (14) $$

Evaluating the right-hand side of (14) includes computing the infinite sum that multiplies the periodic array coefficient. Just as with the infinite series encountered in (10), this series is evaluated using the Kummer transform.

Because the $C_n \to 0$ as $n \to \infty$, a sufficiently large yet finite truncation on $n$ on the left-hand side of (14) provides convergence of the sequence, and the $C_n$ are solved numerically by matrix inversion.

The $C_n$ for $kd = 3$ and $\phi_i = 45^\circ$ are given in Table 3, and the phase difference between adjacent elements is approximately $kd$. This linear phase change suggests that the $C_n$ represent a decaying wave propagating with the free-space phase velocity from the end along the array. The magnitudes $|C_n|$ decrease monotonically, and vary qualitatively, but not precisely, as $n^{-1}$. In $\theta = \phi_i$, uniformly spaced samples of a pure cylindrical wave decay as $n^{-1/2}$; however, this edge-launched wave loses energy upon collision with each element, resulting in the observed faster decay rate.

The resulting total coefficients $c_n$ are compared to those in the periodic grating in Table 3, and, as expected, quickly approach the value of the periodic coefficient with distance from the edge.

V. DECOMPOSING THE FINITE ARRAY

In Section II, the surface coefficients for the finite array were calculated directly from the Toeplitz matrix equation. But as mentioned earlier, the element coefficients in the center of a large array are essentially those of the periodic grating. In fact, the finite array can be decomposed into a periodic array and two seminfinite arrays—one to account for left edge effects and the other to account for right edge effects. The decomposition is only approximate, because, for small finite arrays, the edges cannot be decoupled. Also, for small angles of plane wave incidence (near grazing), the edge angles interact. However, it will be seen that the decomposition does hold for all but the most extreme cases of array smallness and acute incidence angle.

Hence, the currents on an array of $N$ elements can be approximated by superimposing the left edge effects, the right edge effects, and the periodic array distribution:

$$ a_n = C_n(\phi_i) + C_{N-1-n}(\pi - \phi_i) \times e^{-i k d (N-n) \cos \phi_i} + b_n e^{-i k d \cos \phi_i}. \quad (15) $$

The $C_n(\phi_i)$ are the perturbation coefficients of the left-terminated seminfinite array when the plane wave is incident from the angle $\phi_i$. The phase factor on the second term for the perturbation coefficient of the right-terminated seminfinite array takes account of the phase reference of the plane

### TABLE 1 Coefficients of Central Elements in Finite Array versus Periodic Coefficient $N = 50$, $\phi_i = 90^\circ$, $kd = 3$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$a_n$</th>
<th>$b_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>0.7831 $\angle -121.22$</td>
<td>0.7819 $\angle -121.29$</td>
</tr>
<tr>
<td>23</td>
<td>0.7812 $\angle -121.34$</td>
<td>0.7819 $\angle -121.29$</td>
</tr>
<tr>
<td>24</td>
<td>0.7822 $\angle -121.28$</td>
<td>0.7819 $\angle -121.29$</td>
</tr>
<tr>
<td>25</td>
<td>0.7822 $\angle -121.28$</td>
<td>0.7819 $\angle -121.29$</td>
</tr>
</tbody>
</table>

### TABLE 2 Coefficients of Central Elements in Finite Array versus Periodic Coefficient $N = 50$, $\phi_i = 45^\circ$, $kd = 3$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$a_n$</th>
<th>$b_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>0.6122 $\angle 80.90$</td>
<td>0.6128 $\angle 80.88$</td>
</tr>
<tr>
<td>23</td>
<td>0.6130 $\angle -40.51$</td>
<td>0.6128 $\angle -40.66$</td>
</tr>
<tr>
<td>24</td>
<td>0.6160 $\angle -162.26$</td>
<td>0.6128 $\angle -162.21$</td>
</tr>
<tr>
<td>25</td>
<td>0.6143 $\angle 75.88$</td>
<td>0.6128 $\angle 76.25$</td>
</tr>
</tbody>
</table>

### TABLE 3 Coefficients of Seminfinite and Periodic Array $\phi_i = 45^\circ$, $kd = 3$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$C_n$</th>
<th>$c_n$</th>
<th>$b_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.1303 $\angle 144.61$</td>
<td>0.6261 $\angle -137.20$</td>
<td>0.6128 $\angle -125.18$</td>
</tr>
<tr>
<td>1</td>
<td>0.0615 $\angle -55.34$</td>
<td>0.5528 $\angle 112.02$</td>
<td>0.6128 $\angle 113.27$</td>
</tr>
<tr>
<td>2</td>
<td>0.0398 $\angle 109.74$</td>
<td>0.5952 $\angle 4.89$</td>
<td>0.6128 $\angle 8.27$</td>
</tr>
<tr>
<td>3</td>
<td>0.0291 $\angle -83.07$</td>
<td>0.6331 $\angle 127.90$</td>
<td>0.6128 $\angle -129.81$</td>
</tr>
<tr>
<td>4</td>
<td>0.0227 $\angle 85.22$</td>
<td>0.6337 $\angle 107.83$</td>
<td>0.6128 $\angle 108.65$</td>
</tr>
<tr>
<td>5</td>
<td>0.0185 $\angle 105.79$</td>
<td>0.6121 $\angle 14.63$</td>
<td>0.6128 $\angle 12.90$</td>
</tr>
<tr>
<td>6</td>
<td>0.0156 $\angle 63.74$</td>
<td>0.6121 $\angle 14.63$</td>
<td>0.6128 $\angle 12.90$</td>
</tr>
<tr>
<td>7</td>
<td>0.0134 $\angle 126.35$</td>
<td>0.6044 $\angle 104.99$</td>
<td>0.6128 $\angle 104.02$</td>
</tr>
<tr>
<td>8</td>
<td>0.0116 $\angle 43.73$</td>
<td>0.6185 $\angle -16.58$</td>
<td>0.6128 $\angle -17.52$</td>
</tr>
<tr>
<td>9</td>
<td>0.0102 $\angle 146.10$</td>
<td>0.6230 $\angle -139.18$</td>
<td>0.6128 $\angle -139.07$</td>
</tr>
<tr>
<td>10</td>
<td>0.0092 $\angle 24.23$</td>
<td>0.6153 $\angle 98.56$</td>
<td>0.6128 $\angle 99.39$</td>
</tr>
</tbody>
</table>
wave that is now effectively incident from the angle $\pi - \phi_i$.

The standing-wave nature of the $a_n$ coefficients for the finite
array suggests that such a superposition of effectively traveling
waves is appropriate. Using this method, the $a_n$ are

In Figure 2, the coefficients are computed using both
methods for an array of 20 wires with $\phi_i = 45^\circ$ and $kd = 3$.
As the graphs show, the superposition almost perfectly repro-
duces the true $a_n$. This result is reasonable because the array
is large and the edges behave independently—an assumption
made when superimposing two seminfinite arrays. With an
incidence angle of $5^\circ$ (Figure 3), however, the superposition
does not provide highly accurate values for the array coeffi-
cients. This phenomenon occurs because, for an extremely
shallow $\phi_i$, the existence of the right edge is felt by the left
edge. In essence, the sharp angle of incidence puts the left
dege in the shadow of the right edge. This shadowing effect is
evidenced by the worsening of the coefficient agreement
toward the left edge of the array.

Figures 4 and 5 repeat the above example for an array of
five wires. As Figure 4 suggests, superposition is applicable to

an array of even relatively few elements, indicating that for
$\phi_i = 45^\circ$ the multiple edge interactions are still negligible.
When $\phi_i = 5^\circ$ (Figure 5), the edge interaction is even greater
with 5 elements than with 20. Of course, with the smaller
array, the edges will strongly couple, and the superposition
finally fails to accurately model the array.

For the closer spacing $kd = 1$, with $\phi_i = 45^\circ$ and $N = 20$,
the superposition algorithm performs superbly (Figure 6), as
it did for the similar case when $kd = 3$. This invariance to $kd$
dicates that the number of elements is more important,
with respect to edge interaction, than the actual size of the
array. (With $kd = 3$, the array is physically three times larger
than with $kd = 1$.) This fact implies that the elements within
the array impede edge coupling, not the distance between the
edges. Comparing Figure 7 to Figure 4 confirms this notion.
The change in the size of the array by a factor of 3 has not
affected the accuracy of the superposition, even for just five
elements. If $\phi_i = 90^\circ$, the influence of the incident wave on
dege coupling is minimized, and the coupling is a factor of
array size only. In such cases of broadside incidence, the
superposition approach holds for arrays of as few as three
elements.
wave on the finite array. This superposition is shown to hold for even small arrays and the most acute angles of incidence. In contrast to the direct inversion of the Toeplitz matrix and related purely numerical techniques that solve the entire finite array problem all at once, the edge-wave decomposition provides a glimpse into the physics of the wave interactions in a finite array.

APPENDIX A
EVALUATING THE SERIES $\sum_{n=0}^{\infty} e^{i n a} / \sqrt{n}$

The infinite series $S = \sum_{n=0}^{\infty} e^{i n a} / \sqrt{n}$ is evaluated by first writing it in terms of its $N$th partial sum plus remainder as

$$S = \sum_{n=1}^{N} \frac{e^{i n a}}{\sqrt{n}} + \sum_{n=1}^{\infty} \frac{e^{i (N+n) a}}{\sqrt{n + n}}.$$  \hspace{1cm} (A.1)

The finite series is directly summed, and the second infinite series is approximated as a sum of weighted gamma functions by the following method.

Writing the integral form of the gamma function with $nx$ as the variable of integration, where $n$ is a constant integer, gives an integral representation for powers of $n$ as [9]

$$\frac{1}{n^z} = \frac{1}{\Gamma(z)} \int_0^\infty e^{-nx} x^{z-1} \, dx \hspace{1cm} (\Re z > 0). \hspace{1cm} (A.2)$$

Substituting this expression into (A.1) with $z = 1/2$, interchanging the summation and integration, and combining like powers yields

$$S' = \frac{e^{i Na}}{\sqrt{\pi}} \int_0^\infty \sum_{n=1}^{\infty} \frac{e^{i (n-a) x}}{\sqrt{x}} x^{-1/2} e^{-Nx} \, dx,$$ \hspace{1cm} (A.3)

where $S'$ is used to denote the remainder series of (A.1). Employing the geometric series representation for the integrand in the above expression gives

$$S' = \frac{e^{i (N+1) a}}{\sqrt{\pi}} \int_0^\infty e^{-x} - e^{-ia} x^{-1/2} \, dx.$$ \hspace{1cm} (A.4)

This integral is of the form

$$f(N) = \int_0^\infty e^{-N x} g(x) \, dx,$$ \hspace{1cm} (A.5)

which is computed by Watson's lemma [10] as

$$f(N) = \sum_{m=0}^{M} g_m \Gamma(\lambda + m + 1) N^{-(\lambda + m + 1)} + O(N^{-(\lambda + M + 2)}).$$ \hspace{1cm} (A.6)

The $g_m$ are the coefficients of the Taylor series expansion of $g(x)$ about $x = 0$. In this case, $g(x) = (e^x - e^{ia})^{-1}$ and then

$$S \approx \sum_{n=1}^{N} \frac{e^{i n a}}{\sqrt{n}} + \frac{e^{i (N+1) a}}{\sqrt{\pi}} \sum_{m=0}^{M} \frac{\Gamma(m + \frac{1}{2})}{N^{(m+1/2)}},$$ \hspace{1cm} (A.7)

where the inverse power series on $m$ converges very rapidly for $N \gg 1$. Three terms are sufficient to provide convergence.

The above examples illustrate that the method of decomposing the array into more fundamental components is a germane procedure for all but the most extreme cases of incidence angle and array size. This eventual breakdown does not detract from the usefulness of the decomposition in analyzing the basic nature of array scattering (and radiation). Valuable information about the finite array can still be gained by examining its periodic and semi-infinite parts.

VI. CONCLUSIONS

Time-harmonic plane wave scattering from finite, infinite, and semi-infinite arrays of thin wires, subject to the Dirichlet boundary condition on their surfaces, is studied to provide insight into the effects of edges and coupling on the total element excitations, which determine the scattered field from a finite array. In the superposition, the finite array currents are written in terms of left edge, right edge, and periodic contributions. Although the edge contributions cannot exist alone, these perturbation currents behave qualitatively as edge-launched waves that propagate down the array at the free-space velocity and combine to form a standing
of the series on \( m \), and the required Taylor coefficients are

\[
\begin{align*}
g_0 &= (1 - e^{i\alpha})^{-1}, \\
g_1 &= -(1 - e^{i\alpha})^{-2}, \\
g_2 &= (1 - e^{i\alpha})^{-3} - \frac{1}{2}(1 - e^{i\alpha})^{-2}. \tag{A.8}
\end{align*}
\]

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LARGE-SIGNAL MODULATION CHARACTERISTICS OF LASER DIODES WITH WEAK COHERENT OPTICAL FEEDBACK

J. Wang, M. K. Haldar, and F. V. C. Mendis
Department of Electrical Engineering
National University of Singapore
10 Kent Ridge Crescent
Singapore 0511

KEY TERMS
Modulation response, semiconductor lasers, optical feedback, subcarrier multiplexing

ABSTRACT
It is shown that although inadvertent weak coherent optical feedback can adversely affect the small-signal response of a directly modulated semi-

1. INTRODUCTION

Direct intensity modulation of semiconductor lasers is widely employed in subcarrier multiplexed (SCM) systems. In these systems, a small amount of light may be unintentionally fed back to the solitary laser diode due to reflections from various discontinuities. Such reflections, although weak, can strongly affect the response of the laser. The effect of weak coherent feedback on the modulation response of lasers has been studied experimentally and theoretically [1, 2]. Theoretical studies have been carried out in the small-signal approximation. Small-signal analysis shows that in the presence of optical feedback, the modulation transfer function will exhibit an undulating form, with the peaks (or troughs) of the transfer function separated by a frequency equal to the reciprocal of the external round-trip time. The peak-to-peak amplitude of the undulations can be quite high. For instance, when a laser diode is modulated with an optical modulation depth (OMD) of 7.1%, the peak-to-peak amplitude can be as high as 10 dB [1]. Small-signal theory [3] also shows that weak optical feedback can adversely affect harmonic and intermodulation distortion in directly modulated laser diodes employed in SCM systems. From the small-signal analysis, it appears that in order to reduce the effect of optical feedback, the feedback level should be stringently controlled to the lowest possible value.

In this article we present the large-signal modulation characteristics of a laser diode by solving its rate equations numerically. The purpose of the article is to show that the large-signal modulation response with weak optical feedback is quite different from the small-signal response and that it is not as badly affected by weak coherent optical feedback. Therefore, weak optical feedback may not be that detrimental to the performance of directly modulated lasers under large-signal conditions in SCM systems.

2. THEORY AND METHOD

A single-mode laser diode with weak coherent optical feedback can be described by the following rate equations [3]:

\[
\frac{dS(t)}{dt} = \left[ g(n,S) - \frac{1}{\tau_p} \right] S(t) + R + 2k_r \sqrt{S(t)S(t-\tau)} \times \cos[\omega_{th} + \phi(t) - \phi(t-\tau)], \tag{1}
\]

\[
\frac{d\phi(t)}{dt} = \frac{\alpha}{2} g[n(t) - n_{th}] - k_r \sqrt{S(t-\tau) \over S(t)} \times \sin[\omega_{th} + \phi(t) - \phi(t-\tau)], \tag{2}
\]

\[
\frac{dn(t)}{dt} = \frac{I(t)}{eV} - \frac{n(t) - g(n,S)}{\tau_e} - g(n,S) - V, \tag{3}
\]

where \( S \) is the photon number, \( \phi \) is the phase of the optical field, \( n \) is the electron density, \( V \) is the volume of active region, \( e \) is the electron charge, \( \tau \) is the roundtrip time of the external cavity, \( \tau_p \) is the photon lifetime, \( \tau_e \) is the carrier lifetime, \( I \) is the injection current, \( \alpha \) is the linewidth enhancement factor, \( R \) is the spontaneous emission rate, \( n_{th} \) is