Electromagnetic Plane Wave Excitation of an Open-Ended, Finite-Length Conducting Cylinder

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Abstract—A mixed boundary value problem is formulated for the surface currents that are induced by a time-harmonic plane wave incident upon an open-ended conducting tube of finite length. Scattered fields are represented by spatial Fourier transforms in the axial dimension for each of the uncoupled azimuthal Fourier modes of this body of revolution. Numerically efficient mathematical expressions having explicit physical significance are derived to solve the set of linear equations from a Galerkin expansion of the currents in terms of Chebyshev polynomials with edge-condition weighting. Resultant surface currents and axial fields are calculated for several combinations of scatterer geometry and frequency, and interpreted partially in terms of travelling waves external and internal to the conducting cylinder.

I. INTRODUCTION

The interaction of a plane wave with a finite-length circular tube is an important boundary value problem in both the electromagnetics and acoustics literature. References [1–19] give an indication of the diversity of techniques and longevity of this and related wave problems. The semi-infinite cylinder is amenable to the classic Wiener-Hopf technique (for example [1]), while a fairly involved modification to treat the cylinder of finite length is described in [2].

Many of the applications ([6,8,9–17]) are concerned with the far-field scattering (radar cross section or RCS) from such a target. A formulation of the boundary value problem as a Fredholm integral equation for the surface currents [20] is the usual foundation for the class of numerical approaches referred to as the “moment method” [11], the “method of weighted residuals” [18], or some related Galerkin technique [21, 22]. Such projection methods are inherently low frequency techniques upon implementation, since the number of required basis functions increases with the electrical size of the cylinder. Often, the equivalence principle in electromagnetics is used to derive an integral equation which employs the free
space Green’s function as the kernel. In acoustic scattering problems, the parallel derivation results in the Kirchoff-Helmholtz integral equation [23]. Subject to Rayleigh’s hypothesis [24], the scattered field everywhere external to the obstacle is written as a sum of outgoing spherical modes in the T-matrix method [9]. This representation of the Green’s function as a sum of normal modes (of free space) is similar in principle to the dual-series representation used in [7, 8]. High frequency studies [12–15] demonstrate some progress toward this important aspect of the cavity-scattering problem. Fourier transformation to the time domain is addressed in [16], [17], and [19].

The present approach to the mixed boundary value problem of Section II is essentially a Galerkin method, wherein the induced surface currents are expanded in terms of full-domain Chebyshev polynomials (in the axial coordinate z) with built-in edge condition behavior. Circular symmetry of this “body of revolution” preserves the orthogonality of the azimuthal modes which are the basis of a trigonometric Fourier series in azimuthal angle φ. The Chebyshev polynomials also constitute a Fourier series in the transformed z coordinate. Advantage is made of the solution to the Helmholtz equation in cylindrical coordinates, to express the axial components of the scattered fields as Fourier integrals in z. This “spectral” approach is a significant departure from integral equation formulations in which the free space Green’s function is written in the general form

$$G(\vec{r}, \vec{r}') = \frac{\exp[ik|\vec{r} - \vec{r}'|]}{4\pi|\vec{r} - \vec{r}'|}$$

Sections II, III, and IV take full advantage of the cylindrical geometry, arriving at linear equation coefficients which display explicit dependence on the geometrical parameters of the physical problem.

The analysis for an incident TM polarized wave is given in Section II. Modifications in the forcing functions and azimuthal mode definitions for TE incidence are presented in Section III. Throughout the development, resultant expressions are obtained with two primary and surprisingly compatible goals. One philosophy behind the analysis is to have the mathematics closely model the physics. Additionally, all of the linear equation coefficients must be in a form suitable for accurate and efficient computation. Both objectives are exemplified in the on-axis scattered field results of Section IV. Computational notes are summarized in Section V. Resultant surface currents and selected on-axis fields are displayed and interpreted in Section VI.

II. THE SCATTERING OF A TM WAVE

A sinusoidal electromagnetic wave of period $2\pi/\omega$ is incident on a hollow, circular cylinder of length $2L$ and radius a (Fig. 1). The cylinder wall is infinitesimally thin and perfectly conducting. With the time factor $\exp[-i\omega t]$ suppressed, Maxwell’s equations in free space are

$$\nabla \cdot \vec{H} = 0 \quad \nabla \cdot \vec{E} = 0 \quad (1)$$

$$\nabla \times \vec{H} = -\frac{i}{\eta} \vec{E} \quad \nabla \times \vec{E} = i\eta \vec{H} \quad (2)$$
where

\[ k^2 = \omega^2 \mu_0 \epsilon_0, \quad \eta^2 = \mu_0 / \epsilon_0 \]  

(3)

and \( \mu_0, \ \epsilon_0 \) are the permeability and permittivity of free space. Choose cylindrical coordinates \((\rho, \phi, z)\) so that the conducting cylinder is at \( \rho = a, \ |z| \leq L. \) Also, with \( x = \rho \cos \phi, \ \ y = \rho \sin \phi, \) define \( \hat{\rho}, \ \hat{\phi}, \ \hat{z}, \ \hat{y}, \) and \( \hat{z} \) to be unit vectors associated with these polar and Cartesian coordinates.

![Figure 1. Plane wave incident upon a hollow tube.](image)

Elimination of \( \overline{E} \) or \( \overline{H} \) from (1) and (2) yields the wave equation

\[ (\nabla^2 + k^2)(\overline{H}, \overline{E}) = (0, 0) \]  

(4)

which must be satisfied by both the incident fields \((\overline{H}^i, \overline{E}^i)\) and the scattered fields \((\overline{H}^s, \overline{E}^s)\). Suppose that a TM wave, polarized with the magnetic field parallel to \( \hat{y}, \) is incident from the direction that makes an angle \( \theta \) with \( \hat{z}. \) Symmetry allows \( \theta \) to be restricted to range from 0 to \( \pi / 2. \) Then

\[ \overline{H}^i = \overline{E}^i \eta^{-1} \hat{y} \exp[-ik(x \sin \theta + z \cos \theta)] \]  

(5)

\[ \overline{E}^i = \overline{E}^i (\hat{z} \sin \theta - \hat{\rho} \cos \theta) \exp[-ik(x \sin \theta + z \cos \theta)] \]

\[ = \overline{E}^i (\hat{z} \sin \theta - \rho \cos \theta \cos \phi + \phi \cos \theta \sin \phi) \exp[-ik(\rho \cos \phi \sin \theta + z \cos \theta)] \]  

(6)

in which the components \( E^i_z \) and \( E^i_\phi \) of \( \overline{E}^i \) are respectively even and odd functions of \( \phi. \) Hence the scattered electric and magnetic fields, \( \overline{E}^s \) and \( \overline{H}^s, \) have Fourier expansions of the form

\[
\begin{pmatrix}
E^s_z \\
E^s_\phi
\end{pmatrix} = \begin{pmatrix}
E^s_0 \\
E^s_0
\end{pmatrix} + 2 \sum_{n=1}^{\infty} \begin{pmatrix}
E^s_n \\
E^s_n
\end{pmatrix} \cos n\phi, \quad E^s_\phi = -2i \sum_{n=1}^{\infty} E^s_{n\phi} \sin n\phi
\]

\[
\begin{pmatrix}
H^s_z \\
H^s_\phi
\end{pmatrix} = -2i \sum_{n=1}^{\infty} \begin{pmatrix}
H^s_n \\
H^s_n
\end{pmatrix} \sin n\phi, \quad H^s_\phi = H^s_0 + 2 \sum_{n=1}^{\infty} H^s_{n\phi} \cos n\phi
\]

(7)

(8)

The perfectly conducting property of the cylinder requires that the \( \hat{z} \) and \( \hat{\phi} \) components of the total electric field \((\overline{E}^i + \overline{E}^s)\) vanish thereon. To achieve this, it is necessary to use, together with its \( \phi \)-derivative, the expansion

\[ e^{-ik\rho \sin \theta \cos \phi} = J_0(k\rho \sin \theta) + 2 \sum_{n=1}^{\infty} (-i)^n J_n(k\rho \sin \theta) \cos n\phi \]  

(9)
in order to identify the Fourier modes of the incident field. The boundary conditions are thus
\[
E_n^s = -E_0^s(-i)^n J_n(ka \sin \theta) e^{-ikz \cos \theta} \sin \theta \quad (n \geq 0) \\
E_n^s = E_0^s(-i)^n J_n(ka \sin \theta) e^{-ikz \cos \theta} \cos \theta \quad (n \geq 1)
\]
(\rho = a, |z| \leq L)

The solution of (4) for the Cartesian components is evidently of the form
\[
E_n^s = E_0^s(-i)^n a \int_{-\infty}^{\infty} d\kappa e^{-i\kappa z} A_n(\kappa) \begin{cases} J_n(\alpha \rho) H_n^{(1)}(\alpha \rho) & (\rho < a) \\
J_n(\alpha \rho) H_n^{(1)}(\alpha \rho) & (\rho > a) \end{cases}
\]
(11)

\[
H_n^s = E_0^s(-i)^{n-1} \frac{a}{k\eta} \int_{-\infty}^{\infty} d\kappa e^{-i\kappa z} B_n(\kappa) \alpha \begin{cases} J_n(\alpha \rho) H_n^{(1)}(\alpha \rho) & (\rho < a) \\
J_n(\alpha \rho) H_n^{(1)}(\alpha \rho) & (\rho > a) \end{cases}
\]
(12)

where \( \alpha^2 = k^2 - \kappa^2 \), with Re \( \alpha \geq 0 \), and the Hankel function of the first kind ensures only outgoing waves at infinity. Here the extensions from \( \rho < a \) to \( \rho > a \) ensure continuity of \( E_n^s \) and, as seen in (14) below, \( E_n^s \) at \( \rho = a \) for all \( z \).

In this way the number of dimensionless functions to be determined is reduced from four to two. These are the subject of a mixed boundary value problem which is addressed by first noting that discontinuities at \( \rho = a \) of the tangential components of magnetic field are confined to \(|z| \leq L\) and then applying the boundary conditions (10).

Elimination of \( H_\rho \) and \( E_\rho \) from (2) yields
\[
\left( \frac{\partial^2}{\partial z^2} + k^2 \right) \begin{pmatrix} E_n^s \\ H_n^s \end{pmatrix} = \frac{i}{\rho} \frac{\partial}{\partial z} \begin{pmatrix} E_n^s \\ H_n^s \end{pmatrix} + ik \frac{\partial}{\partial \rho} \begin{pmatrix} \eta H_n^s \\ \eta E_n^s \end{pmatrix}
\]
(13)

and hence, from (11) and (12)
\[
E_n^s = E_0^s(-i)^n a \int_{-\infty}^{\infty} d\kappa e^{-i\kappa z} \begin{cases} B_n(\kappa) \begin{cases} J'_n(\alpha \rho) H_n^{(1)}(\alpha \rho) & (\rho < a) \\
J'_n(\alpha \rho) H_n^{(1)}(\alpha \rho) & (\rho > a) \end{cases} \\
J_n(\alpha \rho) H_n^{(1)}(\alpha \rho) \end{cases}
\]
(14)

\[
-\frac{n\kappa}{\alpha^2 \rho} A_n(\kappa) \begin{cases} J_n(\alpha \rho) H_n^{(1)}(\alpha \rho) & (\rho \leq a) \\
J_n(\alpha \rho) H_n^{(1)}(\alpha \rho) \end{cases}
\]

\[
H_n^s = E_0^s(-i)^{n-1} \frac{ka}{\eta} \int_{-\infty}^{\infty} d\kappa e^{-i\kappa z} \frac{1}{\alpha} \begin{cases} A_n(\kappa) \begin{cases} J'_n(\alpha \rho) H_n^{(1)}(\alpha \rho) & (\rho \leq a) \\
J'_n(\alpha \rho) H_n^{(1)}(\alpha \rho) \end{cases} \\
J_n(\alpha \rho) H_n^{(1)}(\alpha \rho) \end{cases}
\]
(15)

The Wronskian of \( J_n, H_n^{(1)} \) then enables certain discontinuities in the scattered field at the conducting cylinder to be deduced from (11), (12) and (15). Thus the
total surface currents

\[ J_\phi = -\left[H_x^s\right]_{a^-}^{a^+} = -2i \sum_{n=1}^{\infty} J_{n\phi} \sin n\phi \]

(16)

\[ J_z = \left[H_z^s\right]_{a^-}^{a^+} = J_{0z} + 2 \sum_{n=1}^{\infty} J_{nz} \cos n\phi \]

have, by comparison with (8), modal contributions given by

\[ J_{n\phi} = -\frac{2(-i)^n}{\pi k\eta} E_0^0 i \int_{-\infty}^{\infty} B_n(\kappa) e^{-i\kappa z} d\kappa \]

(17)

\[ J_{nz} = -\frac{2(-i)^n}{\pi k\eta} E_0^0 i \int_{-\infty}^{\infty} \frac{1}{\alpha^2} \left[ k^2 A_n(\kappa) + \frac{n\kappa}{a} B_n(\kappa) \right] e^{-i\kappa z} d\kappa \]

Since these currents must necessarily be confined to the interval \( |z| \leq L \), the inversion of the Fourier transforms yields, since \( B_0(\kappa) \) is undefined for the TM wave,

\[ B_n(\kappa) = -\frac{i^n k\eta}{4 E_0^0} \int_{-L}^{L} J_{n\phi}(z) e^{i\kappa z} dz \quad (n \geq 1) \]

(18)

\[ \frac{1}{\alpha^2} \left[ k^2 A_n(\kappa) + \frac{n\kappa}{a} B_n(\kappa) \right] = -\frac{i^n k\eta}{4 E_0^0} \int_{-L}^{L} J_{nz}(z) e^{i\kappa z} dz \quad (n \geq 0) \]

It is now expedient to evaluate the \( z \)-integrations by making use of the formulae ([25], §7.355)

\[ \int_{-L}^{L} (L^2 - z^2)^{-1/2} T_m(z/L) e^{i\kappa z} dz = \pi i^m J_m(\kappa L) \quad (m \geq 0) \]

(19)

\[ \frac{1}{L^2} \int_{-L}^{L} (L^2 - z^2)^{1/2} U_m(z/L) e^{i\kappa z} dz = \pi i^m (m + 1) \frac{J_{m+1}(\kappa L)}{\kappa L} \quad (m \geq 0) \]

where \( T_m \), \( U_m \) denote Chebyshev polynomials of the first and second kinds. Since \( J_{n\phi}(z) \) and \( J_{nz}(z) \) are expected to behave like \((L^2 - z^2)^{-1/2}\) and \((L^2 - z^2)^{1/2}\)
respectively at the edges \( z = \pm L \), appropriate expansions are

\[
J_n(z) = \frac{4(-i)^n}{\pi k \eta} \frac{E^{(z)}}{\sqrt{L^2 - z^2}} \sum_{m=0}^{\infty} n f_m (-i)^m T_m(z/L) \quad (n \geq 1)
\]

\[
J_n(z) = \frac{4(-i)^n}{\pi k \eta} \frac{E^{(z)}}{\sqrt{L^2 - z^2}} \sum_{m=0}^{\infty} n g_m (-i)^m U_m(z/L) \quad (n \geq 0)
\]

(20)

whence (19) enables (18) to be simplified to

\[
B_n(\kappa) = -\sum_{m=0}^{\infty} n f_m J_m(\kappa L) \quad (n \geq 1)
\]

(21)

\[
A_n(\kappa) + \frac{n \kappa}{k^2 a} B_n(\kappa) = -\alpha^2 \sum_{m=0}^{\infty} n g_m \frac{J_{m+1}(\kappa L)}{\kappa L} \quad (n \geq 0)
\]

Thus the unknown Fourier transforms, in each azimuthal mode, are expressed as sums of Bessel functions with coefficients to be determined by application of the conditions (10).

Since \( \{U_l(z/L); l \geq 0\} \) and \( \{T_l(z/L); l \geq 0\} \) are complete in \( |z| \leq L \), with weight functions \( (L^2 - z^2)^{-1/2} \) and \( (L^2 - z^2)^{-1/2} \) respectively, conditions (10) are satisfied by substituting (11) and (14) and applying the operators

\[
\frac{1}{L^2} \int_{-L}^{L} (L^2 - z^2)^{1/2} U_l(z/L) \, dz \quad \text{and} \quad \int_{-L}^{L} (L^2 - z^2)^{-1/2} T_l(z/L) \, dz \quad (l \geq 0)
\]

to the respective equations to obtain, after use of (19),

\[
a \int_{-\infty}^{\infty} \frac{J_{l+1}(\kappa L)}{\kappa L} A_n(\kappa) J_n(\alpha a) H_n^{(1)}(\alpha a) \, d\kappa
\]

\[
= -\frac{J_{l+1}(kL \cos \theta)}{k L} J_n(ka \sin \theta) \sin \theta \quad (l \geq 0, n \geq 0)
\]

\[
a \int_{-\infty}^{\infty} J_l(\kappa L) \left[ B_n(\kappa) J_n'(\alpha a) H_n^{(1)}(\alpha a) - \frac{n \kappa}{a^2} A_n(\kappa) J_n(\alpha a) H_n^{(1)}(\alpha a) \right] \, d\kappa
\]

\[
= J_l(kL \cos \theta) n \frac{J_n(ka \sin \theta) \cos \theta}{ka \sin \theta} \quad (l \geq 0, n \geq 1)
\]

(22)

But

\[
J'^{(1)}(\lambda) H_n^{(1)}(\lambda) = \frac{1}{2} \left[ J_{n-1}(\lambda) H_{n-1}^{(1)}(\lambda) + J_{n+1}(\lambda) H_{n+1}^{(1)}(\lambda) \right]
\]

\[-\frac{n^2}{\lambda^2} J_n(\lambda) H_n^{(1)}(\lambda) \quad (n \geq 1)
\]

(23)

and further use of the recurrence relations for Bessel functions, after substitution
of (21), enables (22) to be reduced to the linear systems

\[ \sum_{m=0}^{\infty} n^D_{l,m} \psi_m = -J_{l+1}(kL \cos \theta) J_0(k \sin \theta) \sin \theta \quad (l \geq 0) \]  

\[ \sum_{m=0}^{\infty} \left[ n^C_{lm} f_m + n^D_{lm} \psi_m \right] = -\frac{J_{l+1}(kL \cos \theta)}{kL} J_n(k \sin \theta) \sin \theta \quad (n \geq 1, l \geq 0) \]  

\[ \sum_{m=0}^{\infty} \left[ n^C_{lm} f_m + n^D_{lm} \psi_m \right] = J_l(kL \cos \theta) n^D_{lm} \sin \frac{J_n(k \sin \theta)}{k \sin \theta} \cos \theta \]  

However, in terms of the integral \( n I_{l,m} \) defined by

\[ n I_{l,m} \left( \frac{a}{L}, kL \right) = \int_0^\infty J_l(\lambda) J_m(\lambda) J_n \left( \frac{a}{L} \sqrt{(kL)^2 - \lambda^2} \right) H_n^{(1)} \left( \frac{a}{L} \sqrt{(kL)^2 - \lambda^2} \right) d\lambda \]

\[ = L \int_0^\infty J_l(\kappa L) J_m(\kappa L) J_n(\kappa a) H_n^{(1)}(\kappa a) d\kappa \]  

the coefficients may be written in the form

\[ n^C_{lm} = \frac{n}{(kL)^2} \left[ 1 + (-1)^{l+m+1} \right] n I_{l+1,m} \]  

\[ n^D_{lm} = -\frac{a}{L} \left[ 1 + (-1)^{l+m} \right] \left\{ \frac{n I_{l,m} + n I_{l+2,m} + n I_{l,m+2} + n I_{l+2,m+2}}{4(l+1)(m+1)} \right\} \]

\[ n^C_{lm} = -\frac{a}{L} \left[ 1 + (-1)^{l+m} \right] \left\{ \frac{1}{2}(n-1) I_{l,m} + n I_{l,m} - \frac{n^2}{(ka)^2} n I_{l,m} \right\} \]

\[ n^D_{lm} = \frac{n}{(kL)^2} \left[ 1 + (-1)^{l+m+1} \right] n I_{l,m+1} \]  

Therefore only alternate coefficients are non-zero and hence (24) yields disjoint sets of equations for \( \{g_{2p}; p \geq 0\} \) and \( \{g_{2p+1}; p \geq 0\} \) while, for each \( n \geq 1 \), the two sets of equations (25) can be rearranged into disjoint sets for \( \{f_{2p+1}, n g_{2p}; p \geq 0\} \) and \( \{f_{2p}, n g_{2p+1}; p \geq 0\} \).

Further, (27) shows that values of \( n I_{l,m} \) are required only for \((l+m)\) even.
Integral representations of the two factors in the integrand of (26) are

\[ J_n(\alpha a) H_n^{(1)}(\alpha a) = \frac{2}{\pi} \int_0^{\pi/2} H_0^{(1)}(2\alpha a \sin \xi) \cos 2n\xi \, d\xi \]

and, after setting \( l = m - 2p \),

\[ J_{m-2p}(\kappa L) J_m(\kappa L) = \frac{2}{\pi} (-1)^p \int_0^{\pi/2} J_{2p}(2\kappa L \sin \zeta) \cos[2(m - p)\zeta] \, d\zeta \]

according to §6.681(10) of [25]. The \( \kappa \)-integral is now

\[ (-1)^p \int_0^{\infty} J_{2p}(2\kappa L \sin \zeta) H_0^{(1)} \left[ 2a(k^2 - \kappa^2)^{1/2} \sin \xi \right] \, d\kappa \]

\[ = \frac{2}{\pi i} (-1)^p \int_0^{\infty} \int_0^{\infty} J_{2p}(2\kappa L \sin \zeta) \exp \left[ \frac{ik(4a^2 \sin^2 \xi + \gamma^2)^{1/2}}{(4a^2 \sin^2 \xi + \gamma^2)^{1/2}} \right] \cos \kappa \gamma \, d\gamma \, d\kappa \]

\[ = \frac{2}{\pi i} L \sin \zeta T_{2p}(\gamma/2L \sin \zeta) \exp \left[ \frac{ik(4a^2 \sin^2 \xi + \gamma^2)^{1/2}}{(4L^2 \sin^2 \zeta - \gamma^2)^{1/2}(4a^2 \sin^2 \xi + \gamma^2)^{1/2}} \right] \cos \frac{2L \sin \zeta}{(4a^2 \sin^2 \xi + L^2 \sin^2 \zeta \cos^2 \chi)^{1/2}} \, d\gamma \]

\[ = \frac{1}{\pi i} \int_0^{\pi/2} \exp \left[ \frac{2ik(a^2 \sin^2 \xi + L^2 \sin^2 \zeta \cos^2 \chi)^{1/2}}{(a^2 \sin^2 \xi + L^2 \sin^2 \zeta \cos^2 \chi)^{1/2}} \right] \cos 2p\chi \, d\chi \]

by use of formulae given in [25] (§6.677 and §6.671) and then substitution for the Chebyshev polynomial \( T_{2p} \). Thus (26) can be simplified, for \((l + m)\) even, to the form

\[ nI_{l,m}(\frac{a}{L}, kL) = \frac{4L}{\pi i} \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\infty} \exp \left[ \frac{2ik(a^2 \sin^2 \xi + L^2 \sin^2 \zeta \cos^2 \chi)^{1/2}}{(a^2 \sin^2 \xi + L^2 \sin^2 \zeta \cos^2 \chi)^{1/2}} \right] \cos 2n\xi \cos((l + m)\zeta) \cos((l - m)\chi) \, d\xi \, d\zeta \, d\chi \] (28)

However, for computational purposes it is advisable to subtract out the "static" integral, given from (26) by

\[ nI_{l,m}(\frac{a}{L}, 0) = \frac{2L}{\pi i} \int_0^{\infty} J_l(\kappa L) J_m(\kappa L) I_l(\kappa a) K_m(\kappa a) \, d\kappa \] (29)

in order to eliminate the singularity in the integrand of (28). In this way, the required integral is expressed as the sum of the triple Fourier coefficient of a regular
function and the integral of the simplest combination of the Bessel functions:

\[
n I_{l,m} \left( \frac{a}{L}, kL \right) = \frac{2}{\pi i} \int_0^\infty J_l(\lambda) J_m(\lambda) I_n(\lambda \frac{a}{L}) K_n(\lambda \frac{a}{L}) d\lambda \\
+ \frac{4}{\pi^3 i} \int_0^{\pi/2} \int_0^{\pi/2} \cos 2n\xi \cos[(1 + m)\xi] \cos[(1 - m)\chi] d\xi d\zeta d\chi
\]

\[
\exp \left[ 2ikL \left( (a/L)^2 \sin^2 \xi + \sin^2 \zeta \cos^2 \chi \right)^{1/2} \right] - 1 \\
\left( (a/L)^2 \sin^2 \xi + \sin^2 \zeta \cos^2 \chi \right)^{1/2} \\
\] (30)

It is interesting to note that (28) is essentially identical to an expansion coefficient in the dual series formulation of Lee [7].

III. MODIFICATIONS FOR AN INCIDENT TE WAVE

Suppose instead that a TE wave is incident from the same direction. Then

\[
E_i^t = E_0^{0t} g_i \exp[-ik(x \sin \theta + z \cos \theta)] \\
H_i^t = -E_0^{0t} \eta^{-1}(\hat{z} \sin \theta - \hat{z} \cos \theta)i \exp[-ik(x \sin \theta + z \cos \theta)]
\] (31) (32)

where the \( i \) factor has been inserted for convenience. The scattered field components then have the opposite polarity in \( \phi \) to those in (7) and (8), i.e.

\[
\left( \begin{array}{c} E_s^z \\ E_s^\rho \end{array} \right) = -2i \sum_{n=1}^\infty \left( \begin{array}{c} E_{s,nz} \\ E_{s,n\rho} \end{array} \right) \sin n\phi, E_s^\phi = E_{s,0\phi}^0 + 2 \sum_{n=1}^\infty E_{s,n\phi}^0 \cos n\phi
\]

(33)

\[
\left( \begin{array}{c} H_s^z \\ H_s^\rho \end{array} \right) = \left( \begin{array}{c} H_{s,nz} \\ H_{s,n\rho} \end{array} \right) \cos n\phi, H_s^\phi = -2i \sum_{n=1}^\infty H_{s,n\phi}^s \sin n\phi
\]

(34)

Correspondingly, (16) is replaced by

\[
J_\phi = -[H_{s,nz}^{a+} = J_{0\phi} + 2 \sum_{n=1}^\infty J_{n\phi} \cos n\phi
\]

(35)

\[
J_z = [H_{s,\phi}^{a+} = -2i \sum_{n=1}^\infty J_{nz} \sin n\phi
\]

and now \( A_0(\kappa) \), instead of \( B_0(\kappa) \), is undefined. The conditions to be satisfied at the conducting cylinder are now

\[
E_{s,n\phi}^0 = E_0^{0t} (-i)^n J_n(ka \cos \theta) e^{-ikz \cos \theta} \quad (n \geq 0) \\
E_{s,nz}^0 = 0 \quad (n \geq 1)
\]

(36)
Thus the only changes in the subsequent analysis are the different inhomogeneous terms and the inclusion of the \( n = 0 \) terms in the opposite sets of equations. The linear systems corresponding to (24) and (25) are therefore

\[
\sum_{m=0}^{\infty} C_{lm}^{(2)} f_m = J_l(kL \cos \theta) J'_0(ka \cos \theta) \quad (l \geq 0) \tag{37}
\]

\[
\sum_{m=0}^{\infty} \left[ n^2 C_{lm}^{(1)} f_m + n D_{lm}^{(1)} g_m \right] = 0 \quad (n \geq 1, \ l \geq 0) \tag{38}
\]

\[
\sum_{m=0}^{\infty} \left[ n^2 C_{lm}^{(2)} f_m + n D_{lm}^{(2)} g_m \right] = J_l(kL \cos \theta) J'_n(ka \cos \theta)
\]

where the coefficients are given by (27).

**IV. THE SCATTERED FIELD ON THE CYLINDER AXIS**

From (11) and (12), the axial components of \( E^a \) and \( H^a \) are given by

\[
(E_{0z}^a)_{\rho=0} = E^{0i} a \int_{-\infty}^{\infty} e^{-ikz} A_0(\kappa) H_0^{(1)}(\alpha a) \, d\kappa \quad \text{(TM mode)} \tag{39}
\]

\[
(H_{0z}^a)_{\rho=0} = 0 \quad \text{(TM mode)}
\]

\[
(E_{0z}^a)_{\rho=0} = 0 \quad \text{(TE mode)}
\]

\[
(H_{0z}^a)_{\rho=0} = E^{0i} \frac{i\alpha}{k\eta} \int_{-\infty}^{\infty} e^{-ikz} B_0(\kappa) H_0^{(1)'}(\alpha a) \, d\kappa \quad \text{(TE mode)} \tag{40}
\]

The only non-vanishing orthogonal components are, from (11), (12), (14), (15) and the \( \phi \)-component of (2),

\[
(E_{1\rho}^a)_{\rho=0} = \left( iE_{1\phi}^a \right)_{\rho=0}
\]

\[
= E^{0i} \frac{a}{2} \int_{-\infty}^{\infty} e^{-ikz} \left[ B_1(\kappa) H_1^{(1)'}(\alpha a) - \frac{\kappa}{\alpha} A_1(\kappa) H_1^{(1)}(\alpha a) \right] \, d\kappa \tag{41}
\]

\[
(H_{1\phi}^a)_{\rho=0} = \left( -iH_{1\rho}^a \right)_{\rho=0}
\]

\[
= E^{0i} \frac{a}{2\eta} \int_{-\infty}^{\infty} e^{-ikz} \left[ \frac{k}{\alpha} A_1(\kappa) H_1^{(1)}(\alpha a) - \frac{\kappa}{k} B_1(\kappa) H_1^{(1)'}(\alpha a) \right] \, d\kappa \tag{42}
\]
Consequently the remaining Cartesian components on the axis are

\[ (E_z^s, E_y^s)_{\rho=0} = 2 \left( E_{1\phi}^s \right)_{\rho=0} (1,0) \]  \quad (TM \ mode)  \tag{37}\]

\[ (H_x^s, H_y^s)_{\rho=0} = 2 \left( H_{1\rho}^s \right)_{\rho=0} (0,1) \]

\[ (E_z^s, E_y^s)_{\rho=0} = 2 \left( E_{1\phi}^s \right)_{\rho=0} (0,1) \]  \quad (TE \ mode)  \tag{38}\]

Consider the integrals to be evaluated after substituting the series (21) into (39)–(42). By invoking some of the argument used to establish (28), it may be shown that

\[ \int_{-\infty}^{\infty} e^{-ikz} J_m(\kappa L) H_0^{(1)}(\alpha a) \, d\kappa = \]

\[ \frac{2}{\pi i} \frac{(-1)^m}{m/2} \int_{-1}^{1} \frac{\exp \left\{ ik \left[ a^2 + (z + \beta L)^2 \right]^{1/2} \right\}}{\left[ a^2 + (z + \beta L)^2 \right]^{1/2}} \frac{T_m(\beta)}{(1 - \beta^2)^{1/2}} \, d\beta \]  \tag{43}\]

Two further integrals can be deduced by differentiating this result with respect to \( z \) or \( a \). Since

\[ H_1^{(1)}(\alpha a) = H_0^{(1)}(\alpha a) - \frac{1}{\alpha a} H_1^{(1)}(\alpha a) \]

the remaining required integral is

\[ \int_{-\infty}^{\infty} e^{-ikz} J_m(\kappa L) \frac{1}{\alpha} H_1^{(1)}(\alpha a) \, d\kappa \]

\[ = -\frac{2}{\pi k} (-1)^m/2 \int_{-1}^{1} \exp \left\{ ik \left[ a^2 + (z + \beta L)^2 \right]^{1/2} \right\} \frac{T_m(\beta)}{(1 - \beta^2)^{1/2}} \, d\beta \]

This may be deduced by integration of \( a \) times (43), followed by careful consideration of the limit \( a \to 0 \).

**V. NUMERICAL IMPLEMENTATION**

Upon computation of the integral \( nI_m(a/L, kL) \) in (30), the linear equation coefficients (27) are known for both TM (24), (25) and TE (37), (38) polarization. The decoupled boundary value problem is solved separately for the \( n \) th azimuthal mode. Truncation to the first five \( (n = 0, 1, 2, 3, 4) \) azimuthal (\( \phi \)) modes and nineteen \((m = 0, 1, \ldots, 18)\) Chebyshev polynomials for the axial \((z)\) variation results in stable current coefficients \( n_{1m} \) and \( n_{2m} \) (20) for the test case \( a/L = 0.1, \ kL = 10 \).
The static integral $\int_0^{x_01} \frac{J_0(\xi) I_0(\frac{a}{L} \xi) K_0(\frac{a}{L} \xi)}{d\xi}$ of (29) is required only for $l + m$ even. This infinite range integral with oscillatory integrand is a diagonally strong matrix function of the indices $l$ and $m$. The logarithmic singularity at the origin for the case $l = m = n = 0$ is subtracted and added, resulting in a smooth integrand for Gaussian quadrature. Restricting attention to the finite interval from the singularity to the first zero $x_{01}$ of the Bessel function $J_0$, the process is

$$\int_0^{x_{01}} \frac{J_0^2(\xi) I_0(\frac{a}{L} \xi) K_0(\frac{a}{L} \xi)}{d\xi}$$

$$= \int_0^{x_{01}} \left[ J_0^2(\xi) I_0(\frac{a}{L} \xi) K_0(\frac{a}{L} \xi) + \ln \frac{a}{2L} \xi + \gamma \right] d\xi - x_{01} \ln \frac{a}{2L} + \gamma + 0.2946456$$

where $x_{01} = 2.4048256$ and Euler's constant is $\gamma = 0.57721566$.

The triple Fourier coefficient part of (30), which does depend on frequency, is computed via the three dimensional FFT algorithm FOURN of [26]. In this way, the coefficients for all values of $I$, $m$, and $n$ are computed at once. The error incurred in approximating a set of continuous Fourier coefficients by the discrete algorithm is proportional to the spectral content of the function beyond the truncation point [27]. A $128 \times 256 \times 256$ FFT computation provides good stability for the desired Fourier coefficients of the test case described above.

A series of numerical integrations using the IMSL [28] Gauss-Kronrod routine QDAG is used for the primary range of the static integrals. The tails are approximated by the asymptotic integration by parts [29] of the asymptotic form

$$J_l(\xi) J_m(\xi) I_n(\frac{a}{L} \xi) K_n(\frac{a}{L} \xi)$$

$$\sim \frac{(-1)^{l+m}}{2\pi^{\frac{a}{L}}} \left\{ (-1)^l \left[ \frac{1}{\xi^2} + \frac{c_1}{\xi^4} \right] + \frac{\sin 2\xi}{\xi^2} + \frac{c_1 \sin 2\xi}{\xi^4} + \frac{c_2 \cos 2\xi}{\xi^3} \right\}$$

with

$$c_1 = \frac{1 - 4n^2}{8(a/L)^2} - \frac{(4l^2 - 1)(4l^2 - 9) + (4m^2 - 1)(4m^2 - 9)}{128}$$

$$c_2 = \frac{2m^2 + 2l^2 - 1}{4}$$

The on-axis field $E_z$ for TM polarization is computed by a Chebyshev-Gauss quadrature [30] applied to the integrals of Section IV. Total on-axis field in the deep interior of the tube is on the order of $10^{-5}$ for the test case ($ka = 1$), relative to an incident electric field of unit amplitude. This gives a measure of the accuracy of the computations, since an exponentially small field exists in the deep interior of this cutoff circular waveguide.

VI. RESULTS

When the physical dimensions and angles are systematically varied, the complete solution for the electrodynamic problem of Fig. 1 is worthy of much display and
study. Selected resultant currents and axial fields for TM polarization are included in Figs. 2–10, for angles of incidence $\theta = \pi/4$ and $\pi/2$, as well as the degenerate case of end-on incidence $\theta = 0$. Two frequencies are considered: the familiar test case $kL = 10$ where the total length is $2L \approx 3\lambda$, and the case $kL = 1.5$, which is approximately the dominant resonance of the pipe having $2L \approx \lambda/2$. Two values of the cylinder aspect parameter are included here; $a/L = 0.1$ and 0.5.

Figure 2. Magnitude of $J_z$ for $kL = 10$, $a/L = 0.1$, $\theta = 0$, $\phi = 0$, TM polarization.

Figure 3. Phase of $J_z$ for $kL = 10$, $a/L = 0.1$, $\theta = 0$, $\phi = 0$, TM polarization.
Magnitude and phase of the $\hat{z}$-directed surface current in the $\phi = 0$ plane for $\theta = 0$ are graphed in Figs. 2 and 3 respectively, where the TM designation refers to the $\hat{z}$-directed electric field (6). Current magnitude is relative to the magnitude of the incident magnetic field $E_0/\eta$, while phase is measured with respect to the zero phase of the incident wave at the coordinate origin (Fig. 1). The tube is long enough ($kL = 10$) to exhibit discernible standing wave behavior of the current, which oscillates about the physical optics value 2. A straight line fit to the central portion of the phase plot of Fig. 3 has a slope equal to that of the incident wave in free space. Far from the cylinder surface, the physical problem is equivalent to a pair of $-\hat{z}$ and $+\hat{z}$-travelling waves in the forward region ($z > L$) and a single $-\hat{z}$-travelling wave beyond the rear ($z < -L$). In the region occupied by the cylinder ($-L \leq z \leq L$), the fields of this equivalent transmission line junction problem also consist of oppositely travelling waves.

![Graph of $J_\phi$](image)

**Figure 4.** Magnitude of $J_\phi$ for $kL = 10$, $a/L = 0.1$, $\theta = 0$, $\phi = \pi/2$, TM polarization.

![Graph of $J_z$](image)

**Figure 5.** Magnitude of $J_z$ for $kL = 1.5$, $a/L = 0.1$, $\theta = \pi/4$, $\phi = 0$, TM polarization.
The nature of these (surface) wave components could be better interpreted with the aid of an auxiliary boundary value problem formulation, perhaps in the spirit of [31]. An interesting feature that is persistent in all graphs of axial current (on the illuminated side) is the higher maxima and lower minima toward the exit end. Waves that originate from the discontinuities at each end are slightly lossy, that is, they lose energy to radiation. However, the $-\hat{z}$-travelling wave is continuously reinforced by the incident plane wave. The effect of superimposing the lossy $+\hat{z}$-travelling wave that originates at the exit end upon the more constant $-\hat{z}$-travelling wave is the greater amplitude variations toward the exit end. Note that this current bunching at the exit end is also present in the lower frequency result of Fig. 5. The circumferential ($\hat{\phi}$-directed) currents (drawn as dashed curves in all of the graphs, for example Figs. 4 and 7), are strongest at the ends of the tube, where the surface current turns and changes direction.

**Figure 6.** Magnitude of $J_z$ for $kL = 10$, $a/L = 0.1$, $\theta = \pi/2$, $\phi = 0$, TM polarization.

**Figure 7.** Current magnitude for $kL = 10$, $a/L = 0.1$, $\theta = \pi/2$, $\phi = \pi/2$, TM polarization; $J_z$: solid, $J_\phi$: dashed.
In the case of broadside TM incidence ($\theta = \pi/2$), the current on the illuminated side ($\phi = 0$) of Fig. 6 is much stronger than the current on the dark side ($\phi = \pi$) of Fig. 8. Under TE excitation, the surface current on the cylinder of electrical girth $ka = 1$ is lower in amplitude than for TM incidence. However, the TE surface currents also tend to oscillate about the physical optics value for electrically fatter cylinders having $ka = 5$ and $kL = 10$. Note the larger amplitude of the surface current for the first resonant length ($kL \approx 1.5$) of Fig. 5 where the incident electric field vector has a substantial $z$-component.

![Graph]

**Figure 8.** Magnitude of $J_z$ for $kL = 10$, $a/L = 0.1$, $\theta = \pi/2$, $\phi = \pi$, TM polarization.

![Graph]

**Figure 9.** Magnitude of on-axis $E_z$ for $kL = 10$, $a/L = 0.1$, $\theta = \pi/4$, TM polarization.
Open-Ended Finite-Length Cylinder

Figure 9 depicts the amplitude of the total $E_z$ field on the $z$-axis, for a TM polarized wave incident from $\theta = \pi/4$. Note that the field decays exponentially inside the cutoff cylinder. The on-axis electric field $E_z$ of Fig. 10 is not zero inside the tube with $ka = 5$, due to the propagation of the TM$_{01}$ and TM$_{02}$ waveguide modes. In several observation planes, for various exciting plane waves, the surface current is smaller for the case $ka = 5$ than for the case $ka = 1$ ($kL = 10$ in both cases) because of the penetration of energy into the interior of the tube. That is, the discontinuity in tangential magnetic field is often less in the presence of cut-on (propagating or nonevanescent) waveguide modes.

![Graph](image)

**Figure 10.** Magnitude of on-axis $E_z$ for $kL = 10$, $a/L = 0.5$, $\theta = \pi/4$, TM polarization.

VII. CONCLUSIONS

Induced vector currents on the hollow tube can exhibit extreme variation over the conducting surface, in both the azimuthal and axial directions. The numerical implementation of this mathematical analysis yields a complete solution for both the circumferential and axial currents, as a function of electrical length (frequency), cylinder aspect ratio, and the polarization and orientation of the incident plane wave. The standing wave nature of the surface currents indicates that the interaction between the exciting plane wave and the tube ends occurs via external travelling waves on the conducting cylinder, and via any internal cut-on waveguide modes.

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